Diffusions for Sinai billiards with flat points

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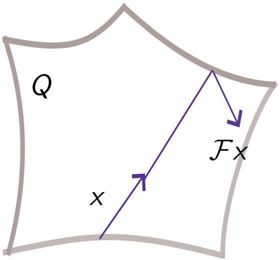
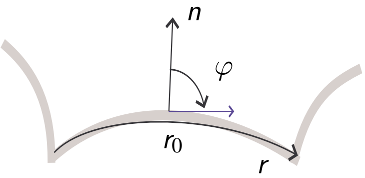
Outline of the talk

- Billiard maps
- Motivations for diffusion properties
- Dispersing billiards with flat points – classical CLT;
- Semi-dispersing billiards with flat points – abnormal CLT;
- Billiards with cusp at a flat point – Stable law.

Classical Sinai billiard map

- Collision space $\mathcal{M} = \partial Q \times [-\pi/2, \pi/2] = \{(r, \varphi)\}$
- Billiard map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$
- Invariant measure $d\mu = (2|\partial Q|)^{-1} \cos\varphi dr d\varphi$.





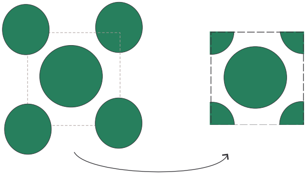
Statistical properties of billiards

- $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is a dispersing billiard map that preserves μ.
- Let $f : \mathcal{M} \to \mathbb{R}^d$ be a nice observable, say $f = \Delta$ being the displacement function in the configuration space, then

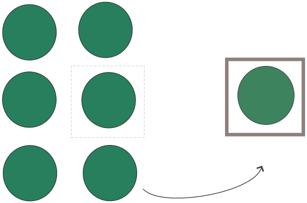
$$X_n := f \circ \mathcal{F}^n$$

  defines a stationary process on $(\mathcal{M}, \mu)$, which are dependent.
- Question: Will the limiting theorems for i.i.d. random variables hold for this stationary process $\{X_n\}$?
- What is $\mathrm{Cov}(X_n, X_0)$? How fast will X_n forget its initial state X_0?

Dispersing billiards on a torus



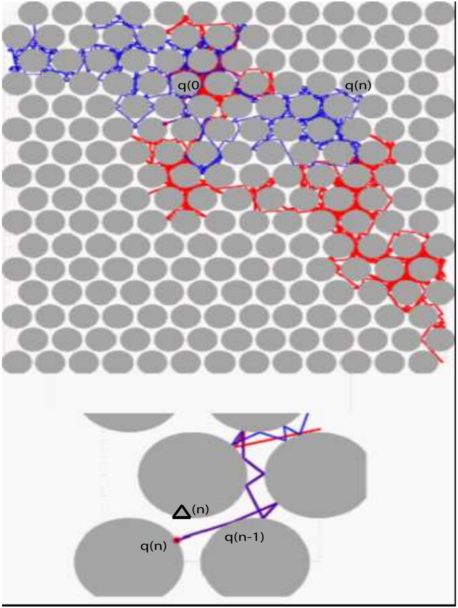
Dispersing billiards with finite horizon



Dispersing billiards with infinite horizon

- Sinai (1960s) etc. proved ergodic, mixing properties.
- Properties of stochastic process $X_n := \Delta \circ \mathcal{F}^n$? Here Δ is the displacement of moving billiard in the unfolding space of Sinai billiard.

Diffusion of Lorentz gas with dispersing scatterers



- $\mathrm{q}(n) = q \circ \mathcal{F}^n$ – position vector;
- $\Delta(n) = \Delta \circ \mathcal{F}^n$ – displacement;
- $\mathrm{q}(n) = \Delta(1) + \cdots \Delta(n)$;
- Ergodicity $\Rightarrow \frac{\Delta(1)+\cdots+\Delta(n)}{n} \to \mathbb{E}(\Delta)$
- Isotropic $\Rightarrow \mathbb{E}(\Delta) \equiv 0$;
- $\mathrm{J} := \lim_{n\to\infty} \frac{\mathrm{q}(n)}{n} = \mathbb{E}(\Delta) = 0$;
- $\Rightarrow$ The diffusion process $\{q(n)\}$ dominated by second term:

$$\mathrm{q}(n) - \mathrm{q}(0) \sim \mathrm{J}\, n + D\sqrt{n}\mathbb{N}(0,1) = D\sqrt{n}\mathbb{N}(0,1)$$

- Question: how to estimate $\mathcal{D}$? finite horizon v.s. infinite horizon

Diffusion of classical Lorentz gas with finite horizon

CLT – Bunimovich, Chernov, Sinai (1991):

$$\frac{\mathrm{q}(n)}{\sqrt{n}} \to \mathcal{D}\mathcal{N}(0,1) \quad \text{in dist.,}$$

where $\mathcal{D}^T \cdot \mathcal{D} = \sum_{n=-\infty}^{\infty} \mathrm{Cov}(\Delta(0)\Delta(n))$

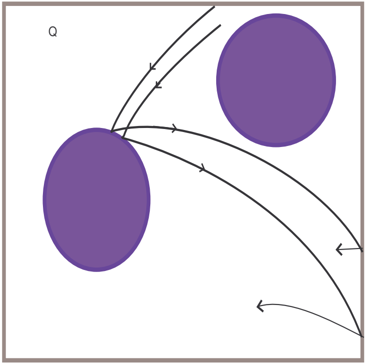
Weak (A.S.) Invariance Principle – Chernov

$$\frac{\mathrm{q}(nt)}{\sqrt{n}} \to \mathcal{D}B_t(x), \text{weakly}, \quad n \to \infty, t \in [0,1]$$

- $\Rightarrow d\mathrm{q}(t) = \mathcal{D}dB_t$ or $\mathrm{q}(t) \approx \mathrm{x} + \mathcal{D}\sqrt{t}N(0,1)$
- $\mathcal{D}$ is determined by speed of correlation functions $\{\mathrm{Cov}(\Delta(0)\Delta(n))\}$ – decay exponentially fast (Young 98).

Green-Kubo formula: $\mathcal{D}^T \cdot \mathcal{D} = \sum_{n=-\infty}^{\infty} \mathrm{Cov}(\Delta(0)\Delta(n))$

Nonequilibrium dispersing billiards



Nonequilibrium phenomena are characterized by the actions on closed Hamiltonian systems including:

- adding external forces;
- a change of temperature;
- adding pressure, etc.

Quantities that can be measured for

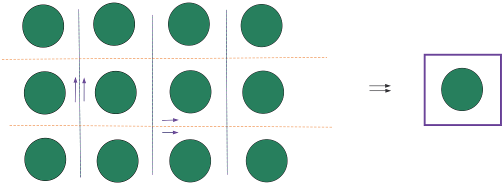
- mass transfer;
- energy (heat) transfer;
- charge transfer (electrical current);
- entropy production, or others.

– Bunimovich, Bonetto, Chernov, Dolgopyat, Eyink, Lebowitz, Pesin, Sinai, more ....

Diffusion for Lorentz gas with infinite horizon – it is enough to study Semi-dispersing billiards

Theorem

The Sinai billiards on a rectangle have mixing rates $\mathcal{O}(\frac{1}{n})$ [Chernov, Zhang, 2005] . The rate is optimal [D. Szasz, T. Varju, 2007].



Remark: $\sum_{n=-\infty}^{\infty} \mathrm{Cov}(\Delta_i \cdot \Delta_j(\mathcal{F}^n)) \sim \sum_{n=-\infty}^{\infty} \frac{1}{|n|} = \infty$

Theorem (Szasz& Varju, 2007)

Semi-dispersing billiards satisfies abnormal CLT: $\frac{q(n)}{\sqrt{n \ln n}} \to DN(0, 1)$.

Diffusion for processes with mixing rates $\mathcal{O}(1/n)$

- P. Bálint and S. Gouezel, Limit theorems in the Bunimovich Stadia, Comm. Math. Phys., 263 (2006), 451–512.
- D. Szasz, T. Varju, Laws and Recurrence for the Planar Lorentz Process with Infinite Horizon. Journal of Statistical Physics, 2007, Volume 129, 59-80.
- N. Chernov and D. Dolgopyat, Anomalous current in periodic Lorentz gases with infinite horizon, Russian Mathematical Surveys, 64 (2009), 651–699.
- P. Bálint, N. Chernov, and D. Dolgopyat, Limit theorems for dispersing billiards with cusps, Comm. Math. Phys. 308 (2011), 479–510.

Research goal I:

Find a unified method to find $\mathcal{D}$ for systems with slow rates of decay of correlations of order $\mathcal{O}(1/n)$.

Estimations on D

- $\mathcal{M}$ — a compact Riemannian manifold with boundary $\partial\mathcal{M}$;
- $\mathcal{S}_1 \subset \mathcal{M}$ — a compact set consisting of at most countably many smooth, connected curves, such that $\partial\mathcal{M} \subset \mathcal{S}_1$;
- F — Nonuniformly hyperbolic map from $\mathcal{M} \setminus \mathcal{S}_1$ onto its image;
- F preserves a mixing SRB measure $\mu_{\mathcal{M}}$ on $\mathcal{M}$, such that

$$|\mu_{\mathcal{M}}(f \circ F^n, g) - \mu_{\mathcal{M}}(f)\mu_{\mathcal{M}}(f)| \sim n^{-1}$$

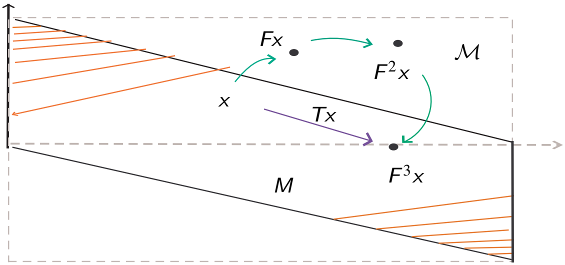
for Holder continuous functions f, g on $\mathcal{M}$.

Question: Does this system satisfy CLT? What is the diffusion coefficient?

Inducing scheme

Return time function

To investigate a nonuniformly hyperbolic system $(F, \mathcal{M}, \mu)$, we find a nice subset $M \subset \mathcal{M}$, and a first return time function $R : M \to \mathbb{N}$ which defines an induced map $T := F^R$ on M that preserves a conditional measure μ_M.



Induced observables

Birkhoff sums:

- Given a smooth observable $f : \mathcal{M} \to \mathbb{R}$, consider the Birkhorff sum for f:

$$S_n(f, F) := f + f \circ F + \cdots + f \circ F^{n-1}$$

- It is enough to consider an induced map (M, T, μ_M), and the induced observable

$$\tilde{f}(x) := f(x) + \cdots + f(T^{R(x)-1}(x))$$

- Birkhorff sum for $\tilde{f}$:

$$S_m(\tilde{f}, T) := \tilde{f} + \tilde{f} \circ T + \cdots \tilde{f} \circ T^{m-1}$$

$S_n f$ vs. $S_n \tilde{f}$

Lemma (Balint, Chernov & Dolgopyat)

If $\{\tilde{f} \circ T^n\}$ satisfies the CLT:

$$\frac{S_n \tilde{f}}{\tilde{\sigma}\sqrt{nH(c_n)}} \xrightarrow{d} N(0,1) \tag{1}$$

then $\{f \circ F^n\}$ satisfies the CLT, with

$$\frac{S_n f}{\sigma\sqrt{nH(c_n)}} \xrightarrow{d} N(0,1), \tag{2}$$

and $\tilde{\sigma}^2 = \sigma^2/\mu(M)$. Here $H(t) := \mathbb{E}(\tilde{f}^2 \mathbb{I}_{|\tilde{f}|<t})$ is a slowly varying function, and $\{c_n\}$ is an increasing sequence (which will be defined later).

Remark: Now it is enough to prove the CLT for $\{\tilde{f} \circ T\}$, but note that $\mathbb{E}(\tilde{f}^2) = \infty$.

Theorem 1 – CLT for $\{X_n := R \circ T^n\}$

Theorem (Mohr&Zhang, 2017)

Under above asumption, also assume $\mathbb{E}(R \circ T \mid R) = \theta R + E$, for $\theta \in [0, 1)$, and $\mathrm{Var}(\sum_{k=1}^{n} E \circ T^k) = \mathcal{O}(n)$. Then the following sequence converges:

$$\frac{X_1 + \cdots + X_n}{\sqrt{(1+\theta)/(1-\theta) \cdot n\mathrm{Var}(X_{n,0})}} \xrightarrow{d} N(0, 1)$$

where $X_{n,k} := (R \cdot \mathrm{I}_{R<\sqrt{n \ln \ln n}}) \circ T^k$.

Difficulty:

- $X_n = R \circ T^n$ has infinite variance;
- Most classical methods on CLT can not be applied.

Martingale approximations

Lemma

There exists a strictly stationary, ergodic martingale difference process array $\{Z_{n,k}, n \geq 1, k = 1, \ldots, n\}$ with respect to the filtration $\{\mathcal{F}_n, n \geq 0\}$ such that $X_{n,k} = Z_{n,k} + Y_{n,k}$, with

$$Z_{n,k} = X_{n,k} - \mathbb{E}(X_{n,k}|\mathcal{F}_{k-1})$$

$$Y_{n,k} := \mathbb{E}(X_{n,k}|\mathcal{F}_{k-1})$$

$$\mathrm{Var}(Z_{n,0}) = (1-\theta)^2 \mathrm{Var}(X_{n,0})$$

We define the σ-algebra $\mathcal{F}_0 = \sigma(\mathcal{R} \circ F^k, k \leq 0)$, and let $\mathcal{F}_{-1}$ be the trivial σ-algebra. We define, for $n \geq 1$

$$\mathcal{F}_n = \sigma(\mathcal{R} \circ F^k, k \in (-\infty, n]) = F^{-1}\mathcal{F}_{n-1}. \tag{3}$$

Then $\mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \mathcal{F}$ and $\mathcal{F}_0$ is the smallest σ-algebra generated by the unstable foliation $\mathcal{W}^u$.

By assumption (A2),

$$Y_{n,k} = \mathbb{E}(X_{n,k}|\mathcal{F}_{k-1}) = \theta X_{n,k-1} + \mathcal{E}_{n,k-1},$$

$$X_{n,k} = Z_{n,k} + Y_{n,k} \Rightarrow (1-\theta)X_{n,k} = Z_{n,k} + \theta(X_{n,k-1} - X_{n,k}) + \mathcal{E}_{n,k-1}.$$

We define the partial sums

$$S_n = \sum_{i=1}^{n} X_{n,i}, \quad M_n = \sum_{i=1}^{n} Z_{n,i}.$$

$$\Rightarrow \quad (1-\theta)S_n = M_n + \theta(X_{n,0} - X_{n,n}) + \sum_{k=0}^{n-1} \mathcal{E}_{n,k}.$$

Since

$$\frac{M_n}{\sqrt{(1-\theta^2)n\mathrm{Var}(X_{n,0})}} \to N(0,1)$$

$$\Rightarrow \quad \frac{S_n}{\sqrt{n\mathrm{Var}(X_{n,0})(1+\theta)/(1-\theta)}} \to N(0,1)$$

In general we have the following central limit theorem.

Theorem

For any observable $f \in H_\gamma$ (Holder on stable manifolds) with $\gamma \in (0,1)$, we assume return time function R satisfy assumptions in above Theorem. Then the following sequence converges in distribution:

$$\frac{f + \cdots + f \circ T^{n-1} - n\mu(f)}{\sqrt{\frac{1+\theta}{1-\theta} \cdot n\mu(M) H(c_n)}} \xrightarrow{d} N(0,1). \tag{4}$$

Remark: In order to apply this theorem to Dynamical Systems to obtain supper diffusion constant, we need to find:

- the precise value of θ:
- a good estimation of $H(c_n) := \mathrm{Var}(\tilde{f} \cdot \mathbb{I}_{R<\sqrt{n \ln n \ln n}})$.

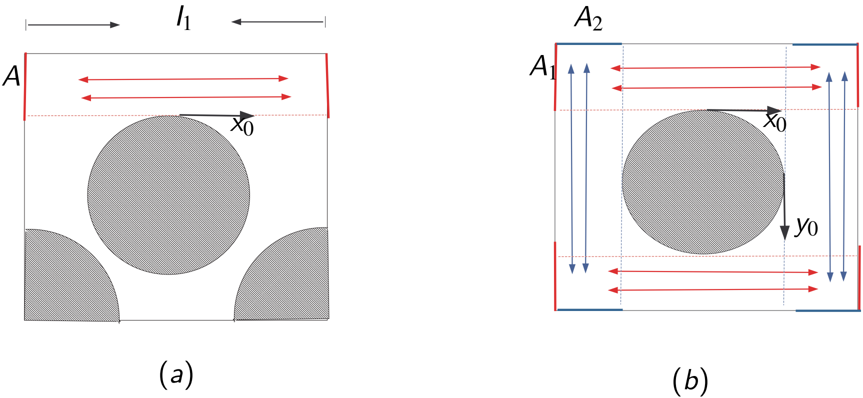


Figure: Two types of semidispersing billiards

Theorem

Let $f \in \mathcal{H}_\gamma$ be Hölder continuous on a small neighborhood of the singular set $\{(r,0) \,:\, r \in A\}$. Assume $\int_{r\in A} f(r,0)\,dr \neq 0$ and the unfolding space of the semidispersing billiards has one channel, then the sequence

$$\frac{f + \cdots + f \circ \mathcal{F}^{n-1} - n\mu(f)}{\sqrt{\sigma_f^2 \cdot n \ln n}} \xrightarrow{d} N(0,1) \tag{5}$$

converges in distribution, as $n \to \infty$, with

$$\sigma_f^2 = \frac{\left(\int_{r\in A} f(r,0)\,dr\right)^2}{4 l_1 |\partial \mathcal{D}|}.$$

Semidispersing billiards with N channels of free flights.

Theorem

Assume the unfolding space of the semidispersing billiards has N channels of free flights. Let $f \in \mathcal{H}_\gamma$ be Hölder continuous on a small neighborhood of the singular set $\{(r, \varphi_k) : r \in A_k, k = 1, \cdots, N\}$. Assume $\sum_{k=1}^N (\int_{r \in A_k} f(r, \varphi_k)\, dr)^2 \neq 0$, then the sequence

$$\frac{f + \cdots + f \circ \mathcal{F}^{n-1} - n\mu(f)}{\sqrt{\sigma_f^2 \cdot n \ln n}} \xrightarrow{d} N(0, 1)$$

converges in distribution, as $n \to \infty$, with

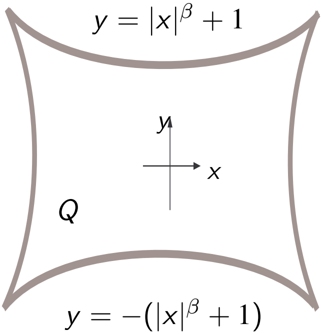
$$\sigma_f^2 = \sum_{k=1}^N \frac{(\int_{r \in A_k} f(r, \varphi_k)\, dr)^2}{4\mathrm{I}_i |\partial \mathcal{D}|},$$

where I_i is the length of the i-th channel in the rectangle.

Research goal II:

- Can one construct a physically meaningful billiard system, with arbitrarily slow decay rates of correlations, of order $\mathcal{O}(n^{-a})$, with $a \in (0, 1)$? and what is the diffusion behavior? (as now CLT fails);
- Main idea: we would like to add some flat points on the boundary of the billiard table, to change the decay rates of correlations.
- How will the rate of decay of correlation depend on the parameter of the order of the flat points on the boundary of dispersing billiards with finite or infinite horizon?

Dispersing billiards with flat points



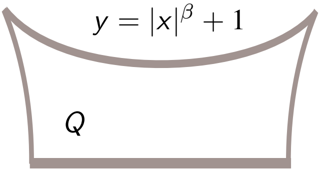
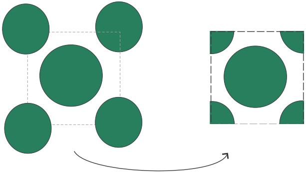
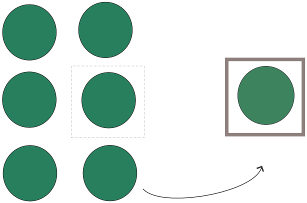


Fig.: Dispersing billiards with walls where the curvature vanishes, $\beta > 2$.



Dispersing billiards with finite horizon



Dispersing billiards with infinite horizon

Remark: We add symmetric flat points for both tables.

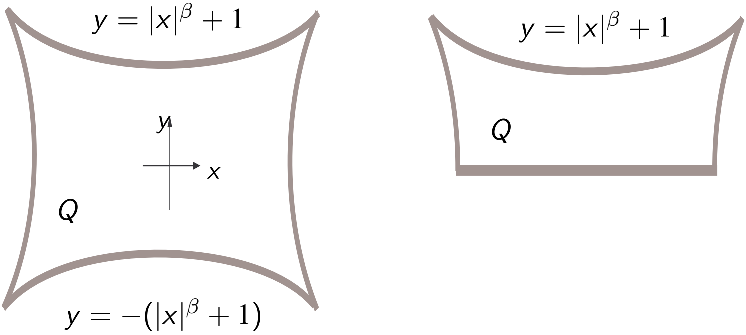


Fig.: There exists a periodic-2 trajectory between two flat points.

Theorem (Chernov & Zhang 2005)

For the dispersing billiards with flat points and finite horizon, the correlations for the billiard map $\mathcal{F}:\mathcal{M}\to\mathcal{M}$ and piecewise Hölder continuous functions f, g on $\mathcal{M}$ decay as

$$|\mathcal{C}_n(f,g,\mathcal{F},\mu)| \leq \text{const}\cdot n^{-1-\frac{4}{\beta-2}}, \quad \beta > 2$$

- If $\beta = 2$, then this is strictly dispersing billiards, which has EDC:

  $$|\mathcal{C}_n(f,g,\mathcal{F},\mu)| \leq C_{f,g}\cdot \vartheta^n,$$

  for some $\vartheta \in (0,1)$.
- When $\beta \to \infty$, this is a semi-dispersing billiards, with $\mathcal{C}_n(f,g,\mathcal{F},\mu) = \mathcal{O}(n^{-1})$.

We fix the scatterer B and label all other copies of the scatterer in the channel as B′, $B_1, \cdots, B_n$, etc. Since the r-coordinate of p is 0, we have $x_0 = (0, \pi/2) \in M$.

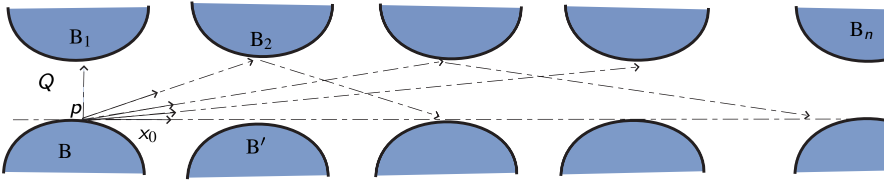


Figure.: Periodic trajectories with collisions only on flat points.

Theorem (Zhang, 2017)

For the dispersing billiards (b) on a torus with flat points and infinite horizon $(\mathcal{F}, \mathcal{M}, \mu)$, if $\beta > 2$, then the correlations for the billiard map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ and $f \in \mathcal{H}^-(\gamma), g \in \mathcal{H}^+(\gamma)$, for $\gamma \in (0, 1)$, decay polynomially:

$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq C\|f\|^-_{C^\gamma}\|g\|^+_{C^\gamma} \cdot n^{-1-\frac{4}{\beta-2}}, \quad (6)$$

where $C = C(\gamma) > 0$ is a constant.

Remark: The classical CLT holds for the billiard map.

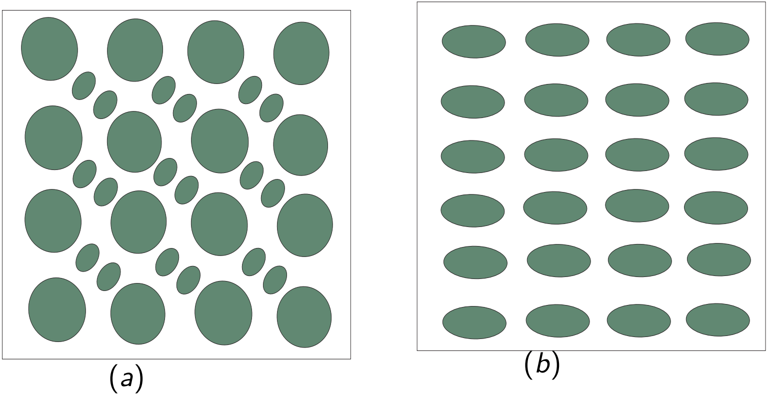


Fig.: Lorentz gas with flat points and finite/infinite horizon.

Remark: What is the diffusion properties?

Semi-dispersing billiards on rectangle table

(h1) $z(s) = -|s|^\beta$, for any $|s| < \varepsilon_1$;

(h2) The tangent line at p is parallel to one straight side of the rectangle R^2.

(h3) For $|s| > \varepsilon_1$, the boundary ∂B belongs to a circular arc with curvature $K \geq 1$.

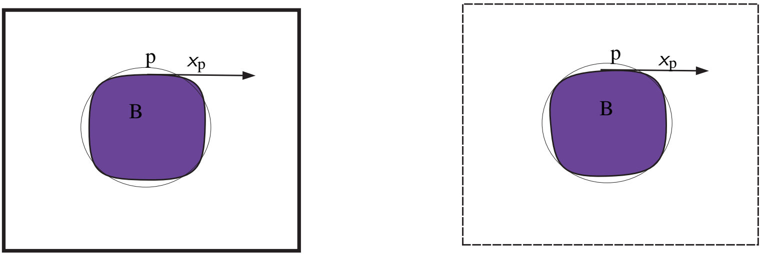
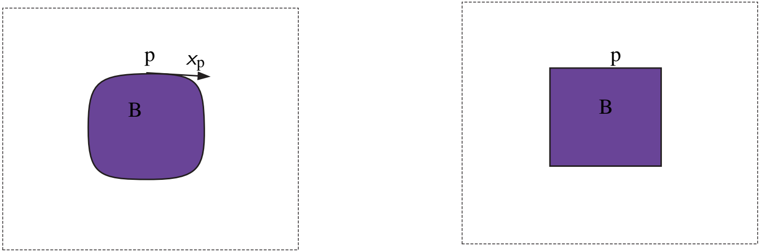


Fig.: (a). Billiards on a rectangle with 4 flat points; (b). Billiards on a torus with 4 flat points.

Semi-dispersing billiards with flat points on a rectangle



(a). Billiards on a rectangle with 4 flat points, for Q_β, the boundaries have zero derivatives up to $\beta - 1$ order at flat points;
(b). The limiting table as $\beta \to \infty$.

- As $\beta \to \infty$, (b) is integrable;
- For $\beta = 2$, the semidispersing billiard has correlation rates $\mathcal{O}(n^{-1})$;
- We guess that as $\beta \in (2, \infty)$, the decay rates should be $\mathcal{O}(n^{-a})$, with $a \in (0, 1)$?

Semi-dispersing billiards on a rectangle

Theorem (Zhang, 2017)

For the family of semidispersing billiards on a rectangle with flat points, if $\beta > 2$, then the correlations for the billiard map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ and piecewise Hölder continuous functions $f \in \mathcal{H}^-(\gamma), g \in \mathcal{H}^+(\gamma)$, for $\gamma \in (0, 1)$, decay polynomially:

$$|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq C\|f\|^-_{C^\gamma}\|g\|^+_{C^\gamma} \cdot n^{-1}, \quad (7)$$

where $C = C(\gamma) > 0$ is a constant.

Remark: The abnormal CLT holds for the difussion process:

$$\frac{q(n)}{\sqrt{n \ln n}} \to \mathcal{D} \cdot N(0, 1)$$

convergence in distribution, as $n \to \infty$.

we can sketch the singular set $\mathcal{S}$ and $\mathcal{F}\mathcal{S}$ in the vicinity of x_0 and $x_1 = \mathcal{F}x_0$.

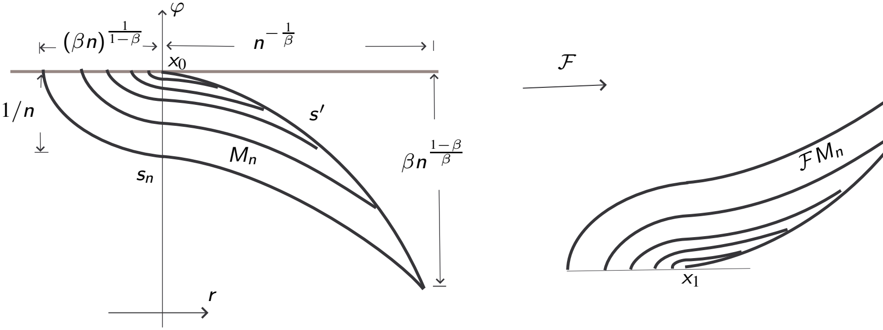


Figure: Singularity curves in $\mathcal{S}$ and $\mathcal{F}\mathcal{S}$ near x_0, for $n \geq n_{\varepsilon_1} := \varepsilon_1^{-\beta}$.

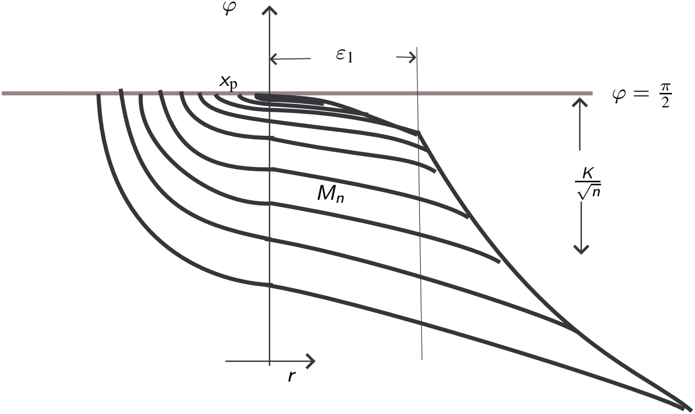


Figure: Singularity curves in $\mathcal{S}_1$ and $\mathcal{F}\mathcal{S}_1$ near x_p, for $1 \leq n \leq n_{\varepsilon_1}$.

Theorem

Fix any $\beta \in [2, \infty)$.

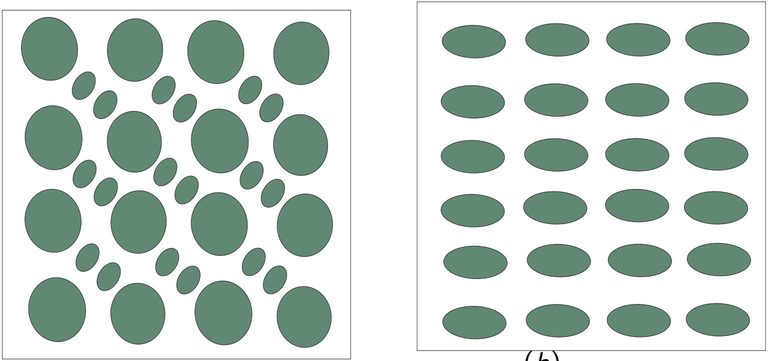
(1) For any $n \geq 1$, the tail bound of the free path satisfies

$$c_1 n^{-2} + \mathcal{O}(n^{-2-\frac{1}{\beta}}) \leq \mu(\tau \geq n) \leq c_2 n^{-2} + \mathcal{O}(n^{-2-\frac{1}{\beta}}) \quad (8)$$

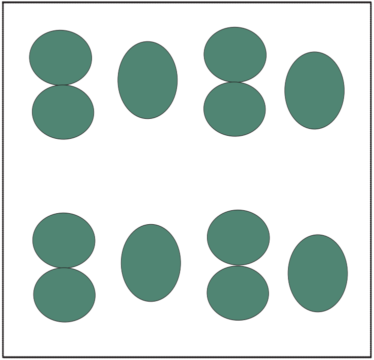
where $c_2 > c_1 > 0$ do not depend on β and n.

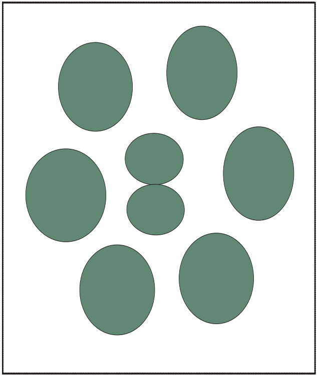
(2) The free path has finite expectation and infinite variance.

Question: Can one construct billiards with flat points such that its decay rates is arbitrarily slow? Say slower than $\mathcal{O}(n^{-1})$?

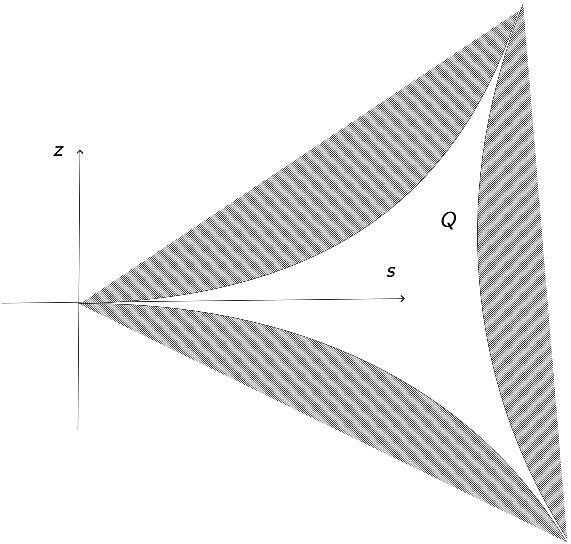


(a) (b)





Consider a billiard table with cusp at the flat point P for $\beta > 2$.



Theorem (Zhang, 2017)

Consider the family of billiards with cusps at flat points on Q_β defined above, with $\beta > 2$. Then for any $\gamma \in (0, 1]$, any observables $f, g \in \mathcal{H}(\gamma)$ on $\mathcal{M}$, there exists $C_{f,g} = C(f, g) > 0$, such that

$$|\mu(f \circ \mathcal{F}^n \cdot g) - \mu(f)\mu(g)| \leq C_{f,g} n^{-\frac{1}{\beta-1}},$$

for $n \geq 1$.

- For the case when $\beta = 2$, the system corresponds to the dispersing billiards with cusps and enjoys mixing rates of order $\mathcal{O}(n^{-1})$.
- Note here $\frac{1}{\beta-1} \in (0, 1)$, which covers all rates slower than $\mathcal{O}(n^{-1})$.

Theorem (Stable Limit Theorem, Paul Jung & Zhang,2017)

Suppose $\mu(f) = 0$ and

$$I_f := \frac{1}{4}\int_0^{\pi} (f(r',\varphi) + f(r'',\varphi)) \sin^{\frac{1}{\alpha}} \varphi \, d\varphi \neq 0$$

where r', r'' are the r-coordinates of the cusp point. Suppose $f \in \mathcal{H}_\gamma$ for some $\gamma > 0$. Then as $n \to \infty$,

$$\frac{\mathcal{S}_n f}{n^{1/\alpha}} \xrightarrow{d} S_{\alpha,\sigma_f}$$

with $\alpha = \frac{\beta}{\beta - 1}$ and $\sigma_f^\alpha = \frac{2I_f^\alpha}{\beta|\partial Q|}$; and S_{α,σ_f} is a stable random variable with characteristic function

$$\mathbb{E}(e^{iuS_{\alpha,\sigma_f}}) = \exp(-|u\sigma|^\alpha(1 - i\text{sign}(u)\tan\frac{\pi\alpha}{2})).$$

Convergence to Levy process

Theorem (J. Paul, F. Pere, Zhang — preliminary results)

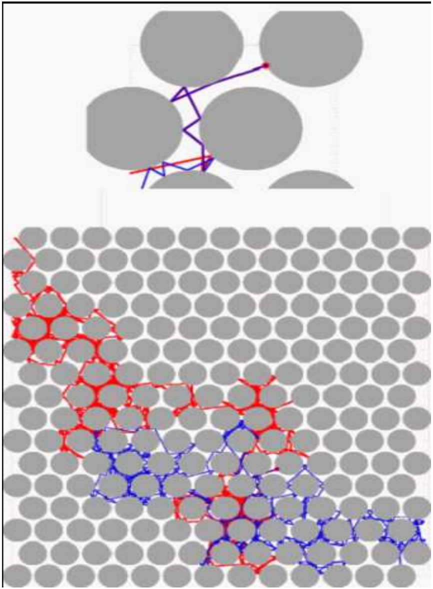
For the billiard map $\mathcal{F}$ with a cusp at a flat point. Let $f : \mathcal{M} \to \mathbb{R}$ be a mean zero Hölder observable and suppose that the value of f at the cusp is positive. Define

$$W_n(t) = n^{-1/\alpha} \sum_{k=0}^{[nt]} f \circ \mathcal{F}^k$$

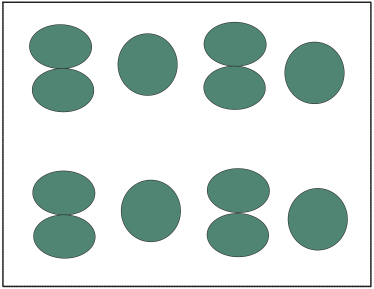
Then W_n converges weakly in the Skorohod M_1 topology to an α-stable Levy process.

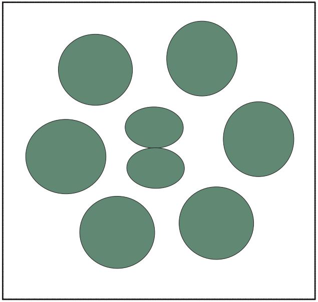
[1] M. Tyran-Kaminska. Weak convergence to Levy stable processes in dynamical systems. Stoch. Dyn. 10 (2010) 263–289.
[2] Ian Melbournea and Roland Zweimullerb, Weak convergence to stable Lévy processes for nonuniformly hyperbolic dynamical systems Annales de l'Institut Henri Poincarr, 2015, Vol. 51, No. 2, 545–556.

Diffusion of Lorentz gases



- With flat points and finite horizon
  – $|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| = \mathcal{O}(n^{-1-\frac{4}{\beta-2}})$
  – classical CLT;
- With flat points and infinite horizon
  – $|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| = \mathcal{O}(n^{-1})$
  – abnormal CLT;
- With flat points at cusps
  – $|\mathcal{C}_n(f, g, \mathcal{F}, \mu)| = \mathcal{O}(n^{-\frac{1}{\beta-1}})$
  – stable law;





- Infinite horizon case;
- Finite horizon case.

For both case, $\lim_{n\to\infty} \frac{f+\cdots+f\circ\mathcal{F}^{n-1}-n\mu(f)}{n^{\frac{1}{\alpha}}} = S_{\alpha,\sigma_q}$ where S_{α,σ_f} is an $\alpha-$ stable variable. i.e $S_n \sim n\mu(f) + n^{\frac{1}{\alpha}} S_{\alpha,\sigma_f}$.

Question: Should the diffusion for the flow dominated by certain Levy process? – maybe not, as the collision time decreases when entering the cusp.

Billiards with infinite domain

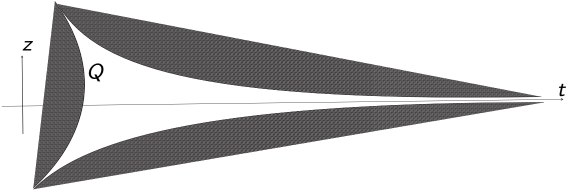


Fig.: A table with an infinite cusp for $\beta \in (2, \infty)$, constructed by Marco Lenci.

Question: What is the diffusion process for the flow? Should it be the Levy process instead of Brownian motion? What is the mixing rates? How to make sense out of the infinite invariant measure?

Theorem (Lenci & Zhang – preliminary results)

For the family of infinite billiards on Q_β defined above, with $\beta > 2$. Then for any $\gamma \in (0, 1]$, any observables $f, g \in \mathcal{H}(\gamma)$ on $\mathcal{M}$, with compact support, then there exists $C_{f,g} = C(f, g) > 0$, such that for $n \to \infty$,

$$n^{\frac{1}{\beta+1}} \mu(f \circ \mathcal{F}^n \cdot g) = \mu(f)\mu(g) + \mathcal{O}(n^{-1+\frac{1}{\beta+1}})$$

Moreover,

$$\lim_{n\to\infty} n^{\frac{1}{\beta+1}} \mu(f \circ \mathcal{F}^n \cdot g) = \mu(f)\mu(g)$$

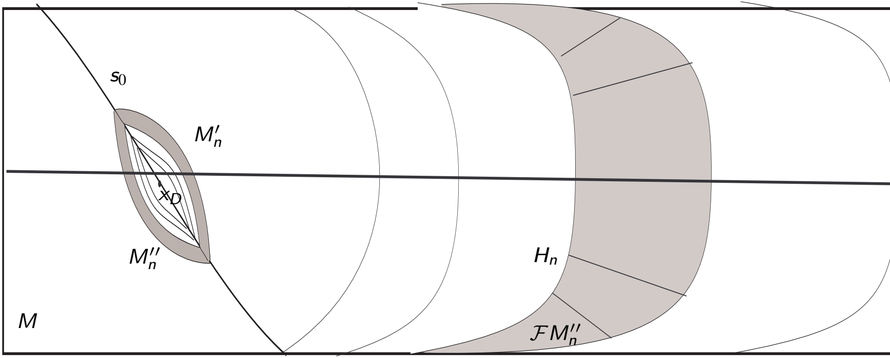


Figure.: Singularity curves of F in the vicinity of x_D; and their forward images under $\mathcal{F}$ near the cusp in $\mathcal{M}$ (that are bounded by H_n).

H_n has equation given by:

$$r = (\sin\varphi)^{\frac{1}{\beta}} C_n^{-\frac{1}{\beta}} + \mathcal{O}\left(\frac{r^{1-2\beta}}{C_n^{1-\frac{1}{\beta}} \sin^{\frac{1}{\beta}} \varphi}\right),$$

where $C_n \sim n^{-\frac{\beta}{\beta-1}}$.

Lemma

For any $N \geq 1$, M_N has measure $\sim N^{-2+\frac{1}{\beta-1}}$. Thus $\mu_M(x \in M : R \geq N) \sim N^{-1+\frac{1}{\beta-1}}$. Moreover, $\mu(x \in \mathcal{M} : R \leq N) \sim N^{\frac{1}{\beta-1}}$

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