Equidistribution for standard pairs in planar dispersing billiard flows

Domokos Szász
(Budapest University of Technology and Economics)
joint with P. Bálint, IP. Tóth (both Budapest),
P. Nándori (College Park),

DinAmici V

Rome, June 2017

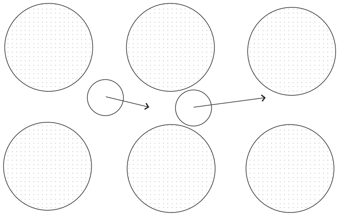
Content

Gaspard-Gilbert, ’08

1. a Hamiltonian - hard disc (or hard ball) - model for the derivation of heat conduction and Fourier law (video). Energy transport but no mass transport!
2. a "promising" two-step strategy by providing the low density limit
3. Step 1: appropriately rescaled energies: $E_1, E_2, \ldots, E_N$ converge to a Markov jump process (rare interaction limit)
4. Step 2: hydrodynamic limit of the Markov jump process provides the heat equation

Difficulty

However, at our present day knowledge even the simplest case: two 2D discs is hopeless (it is a 4D semi-dispersing billiard).



on figure: chain length $N = 2$

periodic scatterers (shaded disks)

confined moving disks (white circles)

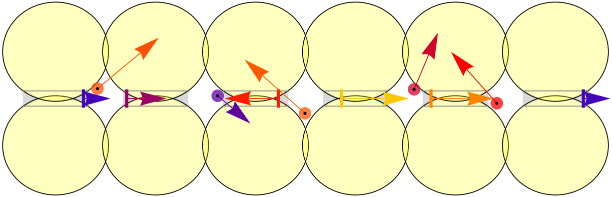
NO mass transport

Ergodicity by Bunimovich-Liverani-Pellegrinotti-Sukhov '92

Parameter choice by Gaspard-Gilbert

Disk-piston model, 2016

T. Gilbert & four of us, '16



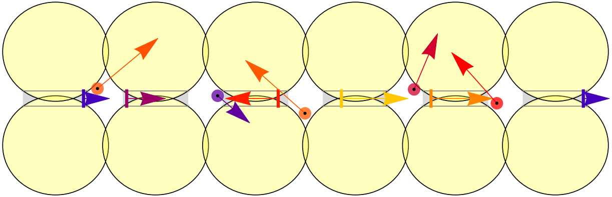
The arrows' lengths and colours reflect the magnitude of kinetic energy of the corresponding disk or piston (in video blends of blue for low energy values, red and yellow for high energy values).

Ergodic (à la Krámli-Simányi-Sz., '91 or Bunimovich-Liverani-Pellegrinotti-Sukhov, '92), however no statistical properties are known.

Movie!

Disk-piston model, 2016

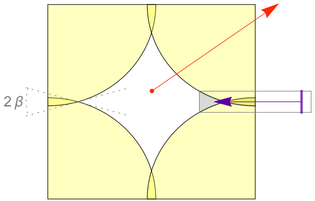
T. Gilbert & four of us, '16

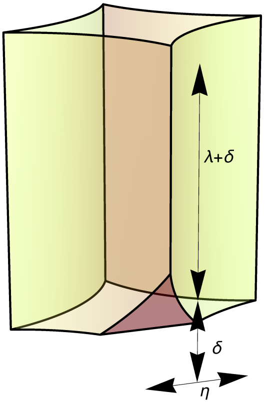


The arrows' lengths and colours reflect the magnitude of kinetic energy of the corresponding disk or piston (in video blends of blue for low energy values, red and yellow for high energy values).

Ergodic (à la Krámli-Simányi-Sz., '91 or Bunimovich-Liverani-Pellegrinotti-Sukhov, '92), however no statistical properties are known.
Movie!

Minimal version: one disk and one piston particle





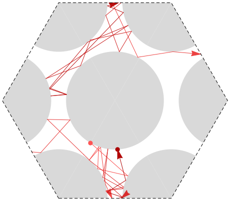
NB: blank diamond $=$
config. space of center of disk
Rare interaction limit $\sim$
$\delta =$ penetration length $\searrow 0$,
$\lambda + 2\delta$ = piston length

isomorphic 3-dim.
semi-dispersing billiard

Method of standard pairs (Dolgopyat-Liverani, '11: smooth Hamiltonian model with weak interaction limit
Need equidistribution statement about the 2D diamond billiard flow starting from a standard pair. (We have a smooth model, too.)

Planar Sinai billiard

Strictly convex scatterers, finite horizon



$M = Q \times S^1$
$Q = \mathbb{T}^2 \setminus \cup_1^N \Gamma_j; \quad \partial\Gamma_j \in C^3$
$\{\Phi^t : M \to M | t \in \mathbb{R}\}$ – the billiard flow
$d\mu = \frac{1}{2\pi Vol(Q)} \mathrm{d}\, r \mathrm{d}\, v$

For $x \in M$, let $\tau(x)$ denote free flight until the first collision:

$$\tau(x) = \inf\{t > 0 \,|\, \pi_Q(\Phi^t(x)) \in \Gamma\}.$$

$$\mathcal{M} = \{(q, v) | q \in \partial Q, |v| \in \mathbb{R}^2, |v| = 1, \langle n(q), v\rangle \geq 0\} \subset M,$$

$$\mathrm{d}\,\nu = \frac{1}{2|\partial Q|}\langle n, v\rangle \mathrm{d}\, q \mathrm{d}\, v$$

$$\Pi x = \Phi^{-\tau_-(x)} x \quad \text{with} \quad \tau_-(x) = \min\{t \geq 0 \mid \Phi^{-t}(x) \in \mathcal{M}\}$$

Hölder properties

- **α-Hölder norm**

$$|f|_{\alpha;H} := \inf\{C \in \mathbb{R} \mid \forall x, y \in X \quad |f(x) - f(y)| \le C dist(x, y)^\alpha\}$$

$$||f||_{\alpha;H} := |f|_{\alpha;H} + \sup_{x \in X} |f(x)|.$$

- **Generalized α-Hölder (semi-)norm** (Keller, '86)

$$|f|_{\alpha;gH} := \sup_{r>0} \frac{1}{r^\alpha} \int_M (osc_r f)(x) d\mu(x).$$

where

$$(osc_r f)(x) := \sup_{y \in B_r(x)} f(y) - \inf_{y \in B_r(x)} f(y).$$

$$var_\alpha(f) := |f|_{\alpha;gH} + \sup_M f - \inf_M f < \infty$$

- **Dymamical θ-Hölder norm** Let $W = W^u$ be an unstable curve for the billiard ball map.

$$|f|_{\theta;dH} := \inf\{C \in \mathbb{R} \,|\, \forall x, y \in W \quad |f(x) - f(y)| \leq C\theta^{s^+(x,y)}\}.$$

where for any $x, y \in W$, their *separation time* $s^+(x, y)$ is the smallest $n > 0$ for which T^n is not continuous on the subcurve of W connecting x and y.

$$||f||_{\theta;dH} := |f|_{\theta;dH} + \sup_W f.$$

Definition

A twice differentiable $W \subset M$ is flow-u-curve if $W := \{\Phi^{t(x)}(x) \mid x \in w\}$ where w is a discrete time u-curve and the flight time function $t : w \to R^+$ satisfies:

(1) Alignement: The angle of the tangent vector of W with the flow direction is at least some $\alpha_{min} > 0$.

(2) Curvature of $W \leq \Gamma_{max}$ everywhere $(0 < \Gamma_{max} < \infty)$.

(3) The length of W is at most some $L_{max} \leq \frac{1}{100\Gamma_{max}}$.

Definition

A flow-standard pair is a pair (W, φ) where

- $W \subset M$ is a flow-u-curve;
- $\varphi : W \to \mathbb{R}^+$ is a probability density function wrt arc length on W ($\tilde{\varphi}$ will denote the measure on W);.
- φ is dynamically Hölder of constants $C_\varphi < \infty$ and $\Theta_\varphi < 1$

Definition

Fix some $\varepsilon_0 \geq 10L_{max}$. We say that W^u is a ε_0-good u-curve if $dist(W^u, \partial M) > \varepsilon_0$.

Let $L = L(W^u)$ be the total length of W^u and $B_{max} = B_{max}(\mathcal{R}_Q)$ an upper bd for the curvature of discrete time u-curves.
So our notion of good u-curve depends on the data

$$\mathcal{R}_u := \{B_{max}, \alpha_{min}, \Gamma_{max}, L_{max}, \varepsilon_0\},$$

and it is assumed that $L \leq L_{max}$ and $10L_{max} \leq \varepsilon_0$.

Theorem (Baladi-Demers-Liverani, 2016)

Consider a billiard like the one introduced above. Assume $0 < \alpha \leq 1$. Then there exist $a' = a'(Q, \alpha) > 0$ and $\mathcal{C}_{BDL} = \mathcal{C}_{BDL}(Q, \alpha) < \infty$ such that for any $F, G : M \to \mathbb{R}$ α-Hölder functions with $\int_M F d\mu = 0$ and any $t \geq 0$ one has

$$\left| \int_M (F \circ \Phi^t) G d\mu \right| \leq \mathcal{C}_{BDL} ||F||_{\alpha;H} ||G||_{\alpha;H} e^{-a' t}.$$

Here $||.||_{\alpha;H}$ denotes the α-Hölder norm.

Corollary (BNSzT)

The validity of the theorem can be extended to the case where both F and G are generalized Hölder functions.

We will only need the case when F is generalized Hölder and G is Hölder.

Theorem (Main)

Let $0 < \Theta_\varphi < 1$, $0 < \alpha_F \leq 1$ *and* $\varepsilon_0 > 0$.
(W^u, φ) *a* $2\varepsilon_0$*-good standard pair with a dyn.* Θ_φ*-Hölder* φ.
Let $F : M \to \mathbb{R}$ *be generalized* α_F*-Hölder.*
Then $\exists \mathcal{C} < \infty$ *and* $\exists a > 0$ *s. t.* $\forall t \geq 0$

$$\left| \int_{W^u} (F \circ \Phi^t)\, \varphi d\, m_{W^u} - \int_M F d\mu \right| \leq \mathcal{C} \cdot ||\varphi||_{\Theta_\varphi; dH} \cdot var_{\alpha_F} F \cdot e^{-at}.$$

Here

- $||\varphi||_{\Theta_\varphi; dH}$ *is the dynamical Hölder norm of* φ;
- $var_{\alpha_F} F$ *is the generalized Hölder seminorm of* F;
- $a = a(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ *and* $\mathcal{C} = \mathcal{C}(Q, \mathcal{R}_u, \Theta_\varphi, \alpha_F) < \infty$ *depend on the billiard table* Q*, the regularity of good u-curves quantified in* $\mathcal{R}_u$*, and the regularity classes of* φ *and* F *given by* Θ_φ *and* α_F*. In other words, they depend on* W^u, φ *and* F *through the aforementioned parameters, only.*

$$\mathcal{R}_Q := \{\tau_{min}, \tau_{max}, \kappa_{min}, \kappa_{max}, \kappa'_{max}, K_{max}, A_{min}, A_{max}, d_Q\}$$

$$\mathcal{R}_u := \{B_{max}, \alpha_{min}, \Gamma_{max}, L_{max}, \varepsilon_0\},$$

Theorem (Main thm continued)

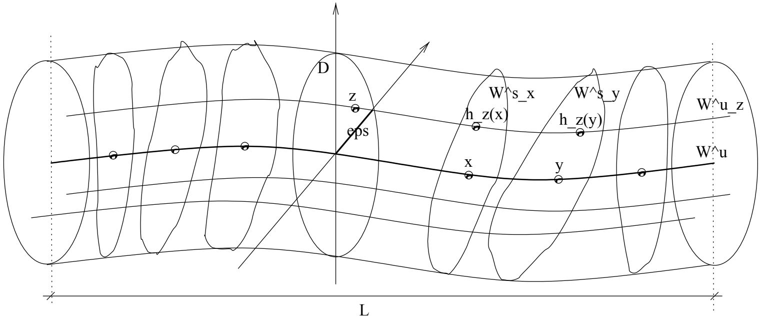
Moreover, $\mathcal{C}$ depends on Q only through $\mathcal{R}_Q$ and, furthermore, through $\mathcal{C}_{BDL}(Q, \alpha)$ from BDL-Theorem with some $\alpha = \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi) > 0$. Similarly, a depends on $(Q, \mathcal{R}_u, \Theta_\varphi)$ only through $a'(Q, \alpha)$ from BDL-Theorem with the same $\alpha = \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi) > 0$. That is,

$$\mathcal{C} = \mathcal{C}(\mathcal{R}_Q, \mathcal{R}_u, \alpha_F, \mathcal{C}_{BDL}(Q, \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi)))$$

and

$$a = a(\alpha_F, a'(Q, \alpha(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi))).$$

The ε-tube



W^u - $2\varepsilon_0$-good u-curve

$D \subset \mathbb{R}^2$ disk of radius $\varepsilon (< \varepsilon_0)$, orthogonal to W^u, centered around mid-point of W^u

L is length of the orthogonal projection of W^u to the normal vector of D

Let ε-tube $U = \cup_{z \in D} W^u_z$ where $W^u_z = W^u + z$

On $U = D \times W^u$ beautiful product structure with invariant (i. e. Lebesgue) measure on it

Lemma

For any $z \in D$ and any $r \in W^u_z$

$$\frac{d\mu_D^{factor}}{d\,m_D}(z)\frac{d\mu^{cond}_{W^u_z}}{d\,m_z}(r) = \frac{1}{Leb(M)}\cos\psi(r).$$

and

$$\frac{d\mu^{cond}_{W^u_z}}{d\,m_z}(r) = \frac{1}{L}\cos\psi(r).$$

m_D is Lebesgue measure on D.
For $r \in U$ $\quad \psi(r)$ is the angle of the tangent vector of W^u_z at r with the normal vector of D.

H = set of those points in W^u whose central-stable manifold crosses U properly, i. e.

1. $W^{c,s}(x)$ is long enough in every direction (meaning $r^{c,s}(x)$ is big enough) so that it surely reaches the boundary of U;
2. $W^{c,s}(x)$ does not hit the circular faces of the tube U, but $W^{c,s}(x)$ rather crosses W^u_z for every $z \in D$.

$$H := \left\{ x \in W^u \;\middle|\; r^{c,s}(x) \geq \frac{10\varepsilon}{\sin c_{tr}} \text{ and } W^{c,s}(x) \cap W^u_z \neq \emptyset \;\; \forall z \in D \right\}$$

Here $c_{tr} = c_{tr}(\mathcal{R}_Q, \mathcal{R}_u) > 0$ is s. t. any W^u and any $W^{c,s}$ intersecting it have an angle at least c_{tr} at $W^u \cap W^{c,s}$

Holonomy. Let $h : H \times D \to U$ be defined so that for any $x \in H$ and $z \in D$ $h(x,z)$ is the single element of $W^{c,s}_x \cap W^u_z$. We will also use the notation $h_z(x) := h(x,z)$, so $h_z : H \to W^u_z$ is the usual holonomy map.

Lemma (classical)

There is a global constant $C_2 < \infty$ such that $m_{W^u}(W^u \setminus H) \leq C_2\varepsilon$.

$$U_0 =: \bigcup_{x\in H} W_x^{c,s} = \{h_z(x) | x \in H, z \in D\} =$$
$$= \bigcup_{z\in D} h_z(H) = \bigcup_{x\in H} \bigcup_{z\in D} \left(W_x^{c,s} \cap W_z^u\right) =$$
$$= h(H \times D)$$

New product structure: For $A \subset H$ and $B \subset D$ we will use the notation $A * B$ to denote this approximate product in U_0, while reserving the notation $A \times B$ for the usual Cartesian product:

$$A * B := h(A \times B) = \bigcup_{x\in A} \bigcup_{z\in B} \left(W_x^{c,s} \cap W_z^u\right).$$

With this notation, we have

$$U_0 = H * D.$$

Construction of the approximating density

$$q(z) = q_\varepsilon(z) := \frac{3}{\varepsilon^2 \pi}\left(1 - \frac{|z|}{\varepsilon}\right) \quad \text{for } z \in D.$$

- q is a prob. density on D wrt m_D, vanishing on ∂D;
- $q(z) \leq \frac{3}{\pi}\frac{1}{\varepsilon^2}$ for any $z \in D$,
- $|q(z_1) - q(z_2)| \leq \frac{3}{\pi}\frac{1}{\varepsilon^3}|z_1 - z_2|$ for any $z_1, z_2 \in D$,

Let $\tilde{q}$ denote the measure on D with density q (wrt m_D)

1. Let $\tilde{G}_0$ be the measure on U_0 defined on approximate product sets $A * B$ as the push-forward of $\tilde{q} \otimes \tilde{\varphi}$ from $H \times D$ to U_0 by h:

$$\tilde{G}_0(A*B) = \tilde{G}_0(h(A\times B)) := (\tilde{q}\otimes\tilde{\varphi})(A\times B)) = \int_A \varphi \mathrm{d}\, m \cdot \int_B q \mathrm{d}\, m_D.$$

2. Let $G_0 := \frac{\mathrm{d}\,\tilde{G}_0}{\mathrm{d}\,\mu} : U_0 \to \mathbb{R}^+$ be the density of $\tilde{G}_0$

$\forall x \in H$, $\forall z \in D$ and $r = h_z(x)$ (for almost every r)

$$G_0(r) = Leb(M)\varphi(x)q(z)\frac{1}{Jh_z(x)}\frac{1}{\cos\psi(r)}.$$

Proposition (Key proposition to smoothing)

There is a constant $1 \le C_\pi = C_\pi(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ *(in fact* $C_\pi = \frac{10}{\sin c_{tr}}$*) such that for any measurable* $A \subset H$*,*

$$\left| \int_{A*D} \left(F \circ \Phi^t\right) G_0 \, d\mu - \int_A \left(F \circ \Phi^t\right) \varphi \, dm \right| \le \int_{A*D} \left[(osc_{C_\pi \varepsilon} F) \circ \Phi^t\right] G_0 \, d\mu.$$

Lemma

If for $f : U_0 \to \mathbb{R}$ $\quad f(r) = f(\pi(r))$ *(i. e.* $f : H \to \mathbb{R}$*), then* $\forall A \subset H$*,*

$$\int_{A*D} (f \circ \pi) G_0 \, d\mu = \int_A f \varphi \, dm.$$

Statements for applicability

Theorem (IP. Tóth)

For any Lebesgue measurable $D \subset \mathbb{R}^d$, *any bounded* $f : D \to \mathbb{R}$, *any* $r > 0$ *and any* $0 < \alpha \leq 1$

$$|osc_r f|_{\alpha;gH} \leq 2(\sup_D f - \inf_D f) Leb(Conv(D)) \left(\frac{2d+1}{r}\right)^{\alpha},$$

where $Conv(D)$ *denotes the convex hull of* D.

Corollary (IP. Tóth)

There is a global constant $C_5 = C_5(\mathcal{R}_Q) < \infty$ *such that for any* $0 < \alpha \leq 1$ *and* $0 < \varepsilon \leq diam(M)$

$$var_\alpha(osc_{C_\pi \varepsilon} F) \leq \frac{C_5}{\varepsilon^\alpha}(\sup_M F - \inf_M F).$$

Regularity of the approximating density

Proposition

1. *If φ is Θ_φ-dyn. Hölder, then G_0 is uniformly* **dyn.** *Hölder when restricted to any element of the u-foliation:* $\exists$ *constants $C_{G;u} = C_{G;u}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ and $\Theta_G = \Theta_G(\Theta_\varphi, \mathcal{R}_Q, \mathcal{R}_u) < 1$ s. t. $\forall\ z \in D$ and $r_1, r_2 \in H_z$*

$$|G_0(r_1) - G_0(r_2)| \leq C_{G;u} \frac{1}{\varepsilon^2} ||\varphi||_{\Theta_\varphi;dH} \Theta_G^{s^+(r_1,r_2)}.$$

($C_{G;u}$ and Θ_G do not depend on φ and ε.)

2. *With $\alpha_{G_0} := \frac{1}{3}$, G_0 is uniformly α_{G_0}-Hölder when restricted to any element of the c, s-foliation:* $\exists$ *constant $C_{G;c,s} = C_{G;c,s}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ s. t. $\forall\ x \in H$ and $\forall\ r_1, r_2 \in W_x^{c,s}$*

$$|G_0(r_1) - G_0(r_2)| \leq C_{G;c,s} \sup_{W^u} \varphi \frac{1}{\varepsilon^3} dist_{W_x^{c,s}}(r_1, r_2)^{\alpha_{G_0}}.$$

Smoothing the approximating density

Proposition (From dyn. Hölder to Hölder)

$\exists = C_6(\mathcal{R}_Q, \mathcal{R}_u) < \infty$, $\alpha_G = \alpha_G(\mathcal{R}_Q, \mathcal{R}_u, \Theta_\varphi) \leq 1$ *and a set* $H_1 \subset H$ *with the following properties: Let* $U_1 = H_1 * D$. *Then*

1. $m_{W^u}(W^u \setminus H_1) \leq C_6\varepsilon$,
2. $\mu(U \setminus U_1) \leq C_6\varepsilon^3$,
3. G_0 *restricted to* U_1 *is* α_G*-Hölder: for any* $r_1, r_2 \in U_1$

$$|G_0(r_1) - G_0(r_2)| \leq C_6 \frac{||\varphi||_{\Theta_\varphi;dH}}{\varepsilon^3} |r_1 - r_2|^{\alpha_G}.$$

Final step: extend $G_0|_{U_1}$ from U_1 to a Hölder $G|_M$

Bound

Lemma

There is a global constant $C_{12} = C_{12}(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ *such that*

$$\left| \int_M (F \circ \Phi^t) G d\mu - \int_M F \circ \Phi^t d\tilde{\varphi} \right| \leq$$

$$\leq \int\limits_M \left[(osc_{C_\pi \varepsilon} F) \circ \Phi^t \right] G d\mu + C_{12} (\sup |F|) (\sup_{W^u} \varphi) \varepsilon,$$

where $C_\pi(\mathcal{R}_Q, \mathcal{R}_u) < \infty$ *if from Key Proposition.*

Questions

1. Extend BDL to billiards with corner points
2. Parameter dependence in BDL
3. Complexity for $3D$ S-billiards
4. Hydrodynamic limit for the disk-piston model