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Modern Trends in the Ergodic Theory of Dynamical Systems

Thermodynamical Formalism for multidimensional intermittent maps

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Consider a compact set X and a smooth map

$$T : X \to X.$$

Denote the set of (ergodic) T-invariant probability measures by

$$\mathcal{M}(T) := \{\mu \colon \mu(T^{-1}A) = \mu(A) \text{ and } \mu(X) = 1\}$$

Definition (Equilibrium measure)

Given a real valued *potential function* $\varphi : X \to \mathbb{R}$. An **equilibrium measure** μ_φ associated to φ is an invariant probability satisfying

$$h_{\mu_\varphi} + \int \varphi \, d\mu_\varphi \quad = \quad \sup_{\nu \in \mathcal{M}(T)} \left\{ h_\nu + \int \varphi \, d\nu \right\} \quad =: \quad P(\varphi)$$

where h_ν denotes the metrical entropy f T.

The equilibrium measure μ_t associated to the **geometric potential**

$$\varphi_t := -t \log |J^u T| \qquad \text{where} \qquad J^u T := DT|E^u(x)$$

- For $t = 0$, the measure μ_0 has maximal entropy
- For $t = 1$ the measure μ_1 is a physical measure, SRB or ACIP.

Question

- *Do there* **exist** *any equilibrium measures associated to a given potential?*
- *Are they* **unique***?*
- *What are their* **statistical properties***, i.e. decay of correlations, central limit theorem?*

The map $T : X \to X$ is called *Anosov* if there exist subspaces $E^s, E^u \subset TX$ and constants $\lambda > 1, c > 0$ such that $TX = E^s \oplus E^u$ and for all $n \in \mathbb{N}$:

$$\begin{aligned} \|DT^n v^s\| &< c\lambda^n \|v^s\| && \text{for } v^s \in E^s, \\ \|DT^{-n} v^u\| &< c\lambda^{-n} \|v^u\| && \text{for } v^u \in E^u. \end{aligned}$$

Theorem (Sinaï, Ruelle, Bowen)

If $T : X \to X$ is Anosov and φ is Hölder continuous, then there exists a unique equilibrium measure associated to the potential function φ. It has good statistical properties.

- Anosov maps are conjugated to the (sub)shift $\sigma(a_0, a_1, a_2, \ldots) := (\not{a_0}, a_1, a_2, \ldots)$
- the set of admissible words of a finite alphabet $\Sigma \subseteq \{1, \ldots, k\}^{\mathbb{N}}$ is compact.
- Any accumulation point of

$$\mu_n = \frac{1}{normalizing} \sum_{a = \sigma^n a} e^{S_n \varphi(a)} \delta_a$$

is an equilibrium measure μ_φ. Here δ_a is the Dirac measure at $a \in \Sigma$ and $S_n\varphi = \sum_{k=0}^{n-1} \varphi(\sigma^k a)$.

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Hyperbolicity is destroyed by

- critical points (quadratic maps)
- homoclinic tangencies (Hénon attractors)
- intermittency (maps with parabolic points: Manneville-Pommeau, Katok).

Katok maps

Consider the Anosov on $\mathbb{T}^2$ with eigenvalues λ, λ^{-1} and eigenvectors s_1 and s_2 given by

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

First perturbation: Slow down near the origin. Let $r_0 \ll 1$ and $0 < \alpha < 1$ and consider an **increasing, convex down** function ψ such that

$$\psi(r) := \begin{cases} \psi(r) = \left(\frac{r}{r_0}\right)^{\alpha} & \text{for} \quad 0 \le r \le r_0/2 \\ \psi(r) = 1 & \text{for} \quad r \ge r_0 \\ \psi'(r) > 0 \text{ and } \psi''(r) < 0 & \text{for} \quad r_0/2 \le r \le r_0 \end{cases}$$

Consider G the time one map of the flow

$$\begin{aligned} \frac{d}{dt} s_1(t) &= \log \lambda \; \psi(r) \; s_1(t) \\ \frac{d}{dt} s_2(t) &= - \log \lambda \; \psi(r) \; s_2(t) \end{aligned}$$

where $r = s_1^2 + s_2^2$.

Remark

Orbits slowed down, but the trajectories of the flow remain unchanged.
The Lyapunov exponent of G at the origin is zero.

Second perturbation: redistribute mass. The slowed down orbits create a singularity of the mass distribution at the origin. Consider the **radial** perturbation

$$\Phi(s_1, s_2) := \frac{1}{\sqrt{k_0}} \left(\int_0^{s_1^2+s_2^2} \frac{dr}{\psi(r)} \right)^{\frac{1}{2}} (s_1, s_2)$$

where $k_0 := \int_{\mathbb{T}^2} \frac{dr}{\psi(r)} < \infty$ since $0 < \alpha < 1$.

Definition

Katok's map is given by

$$G_{\mathbb{T}} := \Phi \circ G \circ \Phi^{-1}.$$

It is a $C^{1+\epsilon}$ for some $\epsilon > 0$ and is C^{∞} outside the origin.

Theorem (Katok)

The Katok map $G_{\mathbb{T}} := \Phi \circ G \circ \Phi^{-1}$ satisfies:

1. *$G_{\mathbb{T}}$ is topologically conjugated to the Anosov A via a homeomorphism h.*
2. *the area m is invariant and ergodic.*
3. *there exist transverse invariant continuous (un)stable directions E^s, E^u and for m almost every x both Lyapunov exponents are non zero.*
4. *δ_0 the Dirac at the origin is the only measure with (two) zero Lyapunov exponents.*
5. *for every $\epsilon > 0$ one can choose r_0 such that*

$$\left| \int \log |J^u G_{\mathbb{T}}| \, dm - \log \lambda \right| < \epsilon.$$

Theorem (Pesin-Senti-Zhang 2017)

Let $G_{\mathbb{T}}$ be the Katok map defined with exponent $0 < \alpha < 1$ and perturbation region $r_0 > 0$ and let

$$\varphi_t := -t \log |J^u G_{\mathbb{T}}|.$$

Then:

1. *$\forall$ $t_0 < 0$ there exists $r_0 > 0$ such that for every $t_0 < t < 1$: there exists a unique equilibrium measure μ_t associated to φ_t. It has exponential decay of correlations and satisfies the Central Limit Theorem with respect to a class which includes all Hölder continuous functions on $\mathbb{T}^2$;*
2. *For $t = 1$ there exist two equilibrium measures associated to φ_1: the Dirac measure at the origin δ_0 and area m.*
3. *For $t > 1$, δ_0 is the unique equilibrium measure associated to φ_t.*

Theorem (Pesin-Senti-Shahidi)

If $\alpha < \frac{1}{4}$ the the area m has polynomial decay of correlations (+ CLT and large deviations).

Corollary

Any surface admits a measure preserving diffeomorphism with polynomial decay of correlations.

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Idea of proof

Take a (small) element $\tilde{P}$ of the Markov partition of the linear Anosov A and consider the first return time

$$R(x) := \min\{n \geq 1 \colon G_{\mathbb{T}}^n(x) \in P = h(\tilde{P})\}.$$

The induced map on $\Lambda = \overline{\bigcup \Lambda_i}$ defined by

$$F|_{\Lambda_i} := G_{\mathbb{T}}^{R_i}$$

is a **Gibbs-Markov-Young induced map**:

(Y1) Markov property;

(Y2) positive leaf volume: $\mu_{\gamma^u}(\gamma^u \cap \Lambda) > 0$ and $\mu_{\gamma^u}((\Lambda \setminus \bigcup \Lambda_i) \cap \gamma^u) = 0$;

(Y3) $F|_{\gamma^s}$ contracts distance, $F|_{\gamma^u}$ expands distance;

(Y4) bounded distortion;

(Y5) integrable return time w.r. μ_γ^u.

Theorem (Pesin-Senti-Zhang, 2016)

Consider a diffeomorphism $f \in C^{1+\varepsilon}(M)$ *of a compact smooth* M *such that* $(Y1)-(Y5)$ *are satisfied. Let* $\varphi_t = -t \log|J^u f|$. *Then*

- *There exists an equilibrium* μ_1 *for* φ_1 *which is an SRB measure (Young 1998).*
- *Assume there exist* $C > 0$ *and* $0 < h < h_{\mu_1}(f)$ *such that*

  $$\sharp\{\Lambda_i \colon R(\Lambda_i) = n\} \leq Ce^{hn}.$$

  Let $\log \lambda_1 := \sup_{\Lambda_i} \sup_{x \in \Lambda_i} \frac{1}{R_i} \log|J^u f^{R_i}|$ *and* $t_0 := \dfrac{h - h_{\mu_1}(f)}{\log \lambda_1 - h_{\mu_1}(f)}$.
  Then for every $t_0 < t < 1$ *there exists a measure* μ_t *which is a unique equilibrium measure for* φ_t. *The supremum is taken over all liftable measures.*
- *If* $GCD(R(\Lambda_i)) = 1$ *and* $\exists K > 0$ *with*

  $$d(f^j x, f^j y) \leq \max\left\{d(x,y), d(Fx, Fy)\right\}$$

  then the measure μ_t *has exponential decay of correlations and satifsifes the central limit theorem w.r. Hölder continuous observables.*

Inducing Schemes

Definition (Gibbs-Markov-Young Inducing Schemes)

An **Inducing Scheme** $\{\Sigma, \tau\}$ is:

- a **countable** collection Σ of disjoint Borel sets $J \subset X$
- a **return time** function $\tau \colon \Sigma \to \mathbb{N}$

such that the **induced map**

$$F|_J := T^{\tau(J)}|_J$$

is semi-conjugated to $\sigma : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ the full shift on a countable alphabet $\Sigma^{\mathbb{N}}$ and that

- the induced map F is hyperbolic
- the induced map F has bounded distortion

Additionally,

- $\limsup_{n\to\infty} \frac{1}{n} \log \sharp\{J \in \Sigma \colon \tau(J) = n\} \leq h < h_{top}(T)$.
- $GCD(\tau_i) = 1$

Thermodynamical formalism on countable alphabets [Mauldin-Urbański], [Sarig].

Theorem (Pesin-S.-Zhang)

Consider a continuous map T of a compact metric space X admitting an inducing scheme. If $\Phi := S_{\tau(x)}\varphi$ is **bounded** *and satisfies*

1. **strongly summable variations**: $\sum_{n\geq 0} nV_n(\Phi) < \infty$
2. **finite Gurevich pressure**: $P_G(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{F^n(x)=x} e^{S_n\Phi(x)} < \infty$
3. **positive recurrence**: *for all* $0 < \delta < \delta_0$ *one has* $P_G\left(\Phi - \tau(P_G(\Phi) - \delta)\right) < \infty$.

There exists a unique equilibrium measure μ_φ associated to φ with:

$$\mu_\varphi := \frac{1}{\nu_\Phi(\tau)} \sum_{k=0}^{\infty} (T^k)_* \nu_\Phi|_{\{\tau\leq k\}}$$

where ν_Φ is the unique F-invariant equilibrium measure associated to Φ.

- strongly summable variations follows from bounded distortion and hyperbolicity.
- finite Gurevic pressure follows from hyperbolicity.
- positive recurrence follows from hyperbolicity and slow growth rate of $S(n)$.
- Use positive recurrence and Abramov and Kac's formulæ (relating the entropy and integral of the induced system to those of the original system) to show that

$$\mu_{\varphi_t} := \frac{1}{\nu_{\Phi_t}(\tau)} \sum_{k=0}^{\infty} (f^k)_* \nu_{\Phi_t}|_{\{\tau \leq k\}}$$

is the unique equilibrium measure.

Other applications: criticality

Dimension one: Quadratic maps: $T_a(x) = x^2 + a$

Theorem (Pesin-S.)

There exists $\mathcal{A}$ with $Leb(\mathcal{A}) > 0$ and $t_- < 0$ and $t_+ > 1$ such that, for any $a \in \mathcal{A}$ the map $T_a = x^2 + a$ admits a unique equilibrium measure μ_t associated to the potential $\varphi_t = -t \log \|DT\|$ for any $t_- < t < t_+$.

μ_t is Bernoulli, has exponential decay of correlations and satisfies the central limit theorem.

Dimension two (homoclinic tangencies): Hénon maps (at the first bifurcation a^*):

$$T_{a,b}(x,y) := (1 - ax^2 + \sqrt{b}z, \pm\sqrt{b}x)$$

Theorem (S.-Takahasi)

For any $t_0 < 0$ there $b_0 > 0$ such that if $0 < b < b_0$, then $T_{a^*,b}$ admits a unique equilibrium measure μ_t associated to $\varphi_t := -t \log \|DT_{a^*,b}|_{E^u}\|$ for all $t_0 < t < 1$.

μ_t has exponential decay of correlations and satisfies the central limit theorem, and is the unique measure of maximal unstable dimension.

Difficulty

Difficulties of the proof:

- Constructing a Young tower with bounded distortion and growth rate condition.
- Liftability: Not every T-invariant probability on X corresponds to a σ-invariant measure of $\Sigma^{\mathbb{N}}$ and vice-versa. First return time easier.

Grazie!