6th June 2017

Quantitative mixing for area-preserving flows on compact surfaces

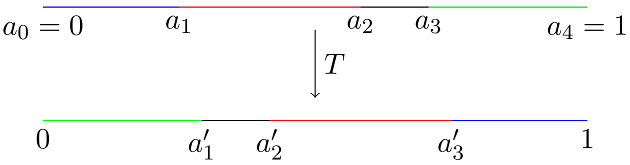
Davide Ravotti

Interval Exchange Transformations

Let $I = [0,1)$. An Interval Exchange Transformation $T\colon I \to I$ of $d \geq 2$ intervals is defined by a permutation $\pi \in \mathcal{S}_d$ and a vector $\underline{\lambda} \in \Delta_d$ as follows

$$\text{let } a_j = \sum_{i \leq j} \lambda_i \text{ and } a'_j = \sum_{i \leq \pi(j)} \lambda_{\pi^{-1}(i)}$$

$$T(x) = x - a_{j-1} + a'_{j-1} \text{ for } x \in [a_{j-1}, a_j).$$



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- Almost every IET is uniquely ergodic (Masur; Veech, '82).
- Every IET is not mixing (Katok, '80).
- If π does not define a rotation, almost every IET is weak mixing (Avila-Forni, '07).

Suspension flows over IETs

Let T be an IET and consider $f \in L^1(I)$, $f > 0$, $\|f\|_1 = 1$. Define

$$\mathcal{X} = \mathcal{X}_f := \{(x, y) \in I \times \mathbb{R} : 0 \leq y \leq f(x)\} /_{\sim},$$

where $\sim$ is the equivalence relation generated by the pairs $\{(x, f(x)),$ $(T(x), 0)\}$. The suspension flow $\{\phi_t\}_{t \in \mathbb{R}}$ over T with roof function f is the unit-speed vertical flow on $\mathcal{X}$.

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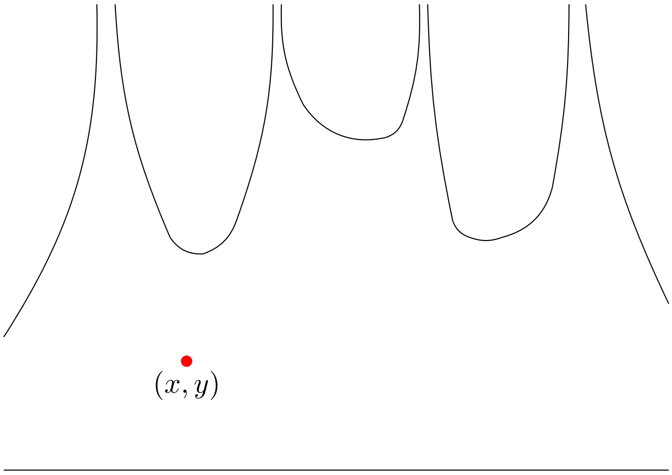
Explicitly, let

$$S_n(f)(x) := \begin{cases} \sum_{i=0}^{n-1} f \circ T^i x & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=-1}^{n} f \circ T^i x & \text{if } n < 0, \end{cases}$$

and $n(x, t) = \max\{n : S_n(f)(x) \le t\}$; then

$$\phi_t(x, 0) = \left(T^{n(x,t)} x, t - S_{n(x,t)}(f)(x)\right).$$

Suspension flows over IETs



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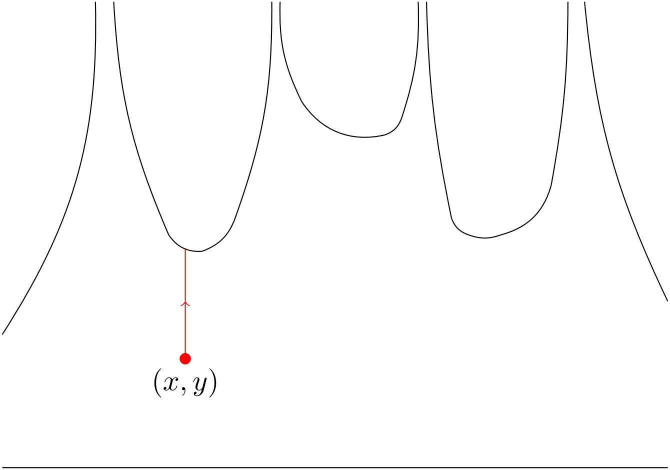

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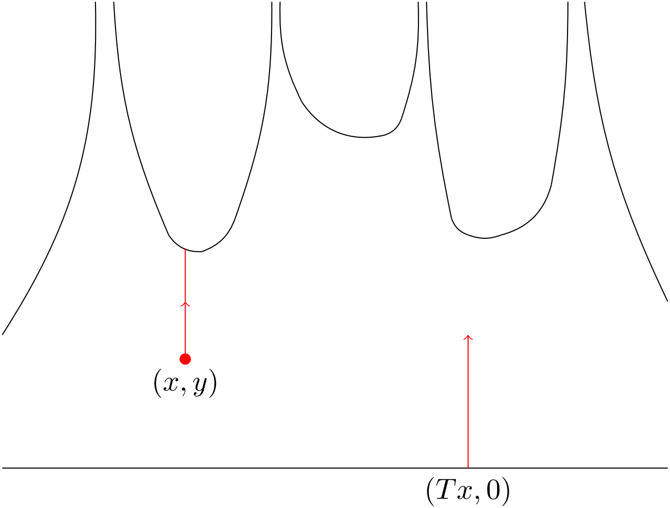

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If f is of bounded variation, then the suspension flow is not mixing (Kochergin, '72; Katok, '80).

We say that f is a roof function with logarithmic singularities if

- $f \in \mathscr{C}^\infty(I \setminus \{a_0, a_1, \ldots, a_d\})$,
- for each $j = 0, \ldots, d$, there exist constants C_j^+, C_j^- and a neighborhood U_j of a_j such that

$$f(x) = C_j^+ \, |\log(x - a_j)| + e(x), \qquad \text{for } x \in U_j, x > a_j,$$
$$f(x) = C_j^- \, |\log(a_j - x)| + \widetilde{e}(x), \qquad \text{for } x \in U_j, x < a_j;$$

where $e, \widetilde{e}$ are smooth bounded functions on I.

The singularities are called symmetric if $\sum_j C_j^+ = \sum_j C_j^-$ and asymmetric otherwise.

Suspension flows over IETs

- For almost every rotation T and for all f with asymmetric logarithmic singularities, the suspension flow $\{\phi_t\}_{t\in\mathbb{R}}$ is mixing (Sinai-Khanin, '92).

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- For almost every IET T and for all f with one asymmetric logarithmic singularity at 0, the suspension flow $\{\phi_t\}_{t\in\mathbb{R}}$ is mixing (Ulcigrai, '07).

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- For almost every IET T and for all f with one asymmetric logarithmic singularity at 0, the suspension flow $\{\phi_t\}_{t\in\mathbb{R}}$ is mixing (Ulcigrai, '07).
- For almost every IET T and for all f with symmetric logarithmic singularities, the suspension flow $\{\phi_t\}_{t\in\mathbb{R}}$ is not mixing (Ulcigrai, '11).

The main result

Theorem (R., '16)

For almost every IET T and for all f with asymmetric logarithmic singularities, the suspension flow $\{\phi_t\}_{t\in\mathbb{R}}$ is mixing. Moreover, for each such $\{\phi_t\}_{t\in\mathbb{R}}$, there exists $0 < \gamma < 1$ such that for all $g, h \in \mathscr{C}_c^1(\mathcal{X})$ with $\int_{\mathcal{X}} g \,\mathrm{d}\,\mathrm{Leb} = 0$, we have

$$\int_{\mathcal{X}} (g \circ \phi_t) h \ \mathrm{d}\,\mathrm{Leb} \leq \frac{C_{g,h}}{(\log t)^{\gamma}},$$

for some constant $C_{g,h} > 0$.

Area-preserving flows on surfaces

Let $\mathcal{M}$ be a compact connected orientable surface equipped with a smooth area form ω.

Each smooth closed 1-form η induces a divergence-free vector field W by contraction: $W \lrcorner \, \omega = \eta$.

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Example

Consider $\mathcal{M} = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ with $\omega = \mathrm{d}x \wedge \mathrm{d}y$; let $a, b \in \mathbb{R}$. Then, if $\eta = a\,\mathrm{d}x + b\,\mathrm{d}y$, we have that $W = b\partial_x - a\partial_y$. In other words, $\{\varphi_t\}_{t\in\mathbb{R}}$ is the linear flow in direction $(b, -a)$.

Area-preserving flows on surfaces

Equip the set $\mathcal{F}$ of smooth closed 1-forms with isolated zeros with the following topology:

$\eta_n \to \eta$ in $\mathcal{F}$ iff for every simply connected open set $U \subset \mathcal{M}$, we can choose H_n, H, with $\eta_n = \mathrm{d}H_n$ and $\eta = \mathrm{d}H$, such that $H_n \to H$ in the $\mathscr{C}^\infty$-norm.

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Generic 1-forms in $\mathcal{F}$ have simple zeros, namely, if $p \in \mathcal{M}$ is a zero of η, then $\det \mathrm{Hes}_p(H) \neq 0$, where $\eta = \mathrm{d}H$ in a neighborhood of p.

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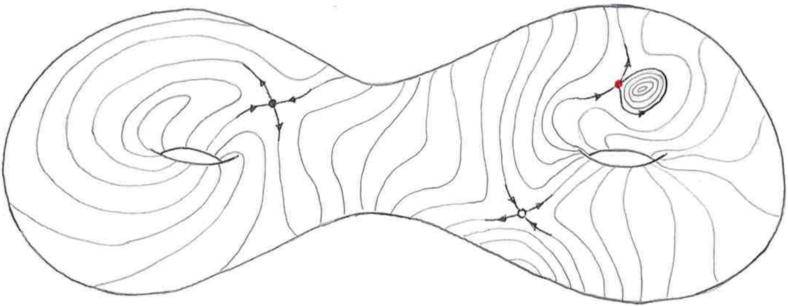
- For each $\eta \in \mathcal{F}$, $\mathcal{M}$ can be decomposed into regions filled with periodic orbits and minimal components (Levitt, '82; Mayer, '43; Zorich, '99).

Area-preserving flows on surfaces



Reduction to suspension flows

Let $\mathcal{M}' \subset \mathcal{M}$ be a minimal component.

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- If $\mathcal{M}' \neq \mathcal{M}$, up to small perturbations we can suppose that the singularities are asymmetric.
- The restriction of the flow induced by η on $\mathcal{M}'$ is isomorphic to the suspension flow over T with roof function f.

Measure class

Let η be a 1-form with l centres and s simple saddles; let $\Sigma(\eta) \subset \mathcal{M}$ be the set of zeros. Choose a basis $\gamma_1, \ldots, \gamma_m$ of $H_1(\mathcal{M}, \Sigma(\eta), \mathbb{R})$, where $m = 2g + l + s - 1$.

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$$\Theta(\eta) = \Big(\int_{\gamma_1} \eta, \dots, \int_{\gamma_m} \eta \Big) \in \mathbb{R}^m.$$

The map Θ is well-defined on a neighborhood of η and the pull-back of the Lebesgue measure on $\mathbb{R}^d$ induces a measure class on the set of 1-forms, sometimes called Katok's measure class.

The main result

Theorem (R. '16)

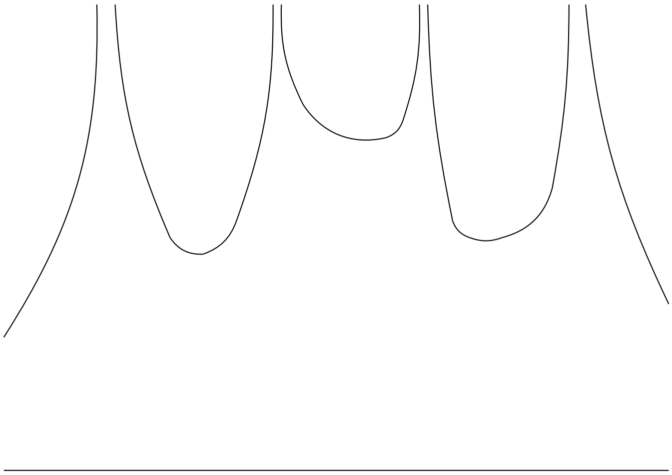
There exists an open and dense subset of closed 1-forms inducing a non-minimal flow such that almost every 1-form in it induces a mixing flow on each minimal component.

For all such $\{\varphi_t\}_{t\in\mathbb{R}}$ and all minimal component $\mathcal{M}' \subset \mathcal{M}$, there exists $0 < \gamma < 1$ such that for every $g, h \in \mathscr{C}^1_c(\mathcal{M}')$ with $\int_{\mathcal{M}'} g\,\omega = 0$ we have

$$\int_{\mathcal{M}'} (g \circ \varphi_t) h\,\omega \leq \frac{C_{g,h}}{(\log t)^{\gamma}},$$

for some constant $C_{g,h} > 0$.

Idea of the proof



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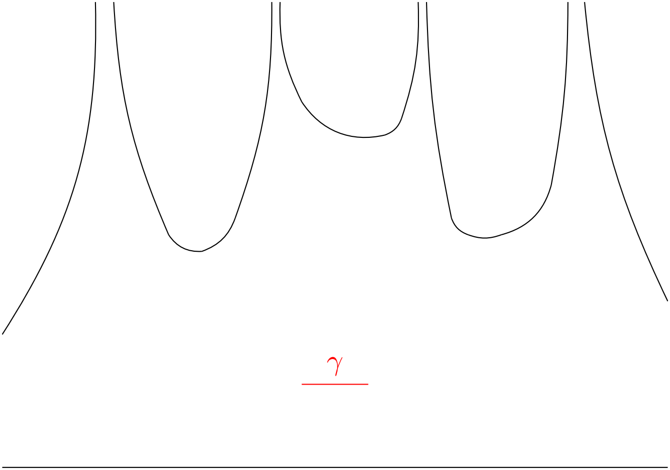

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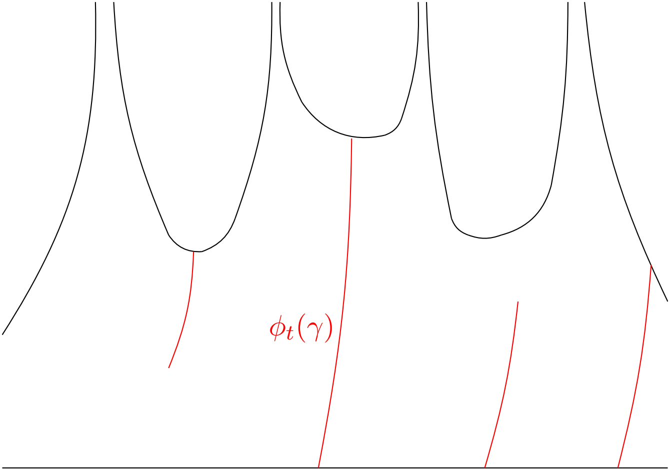

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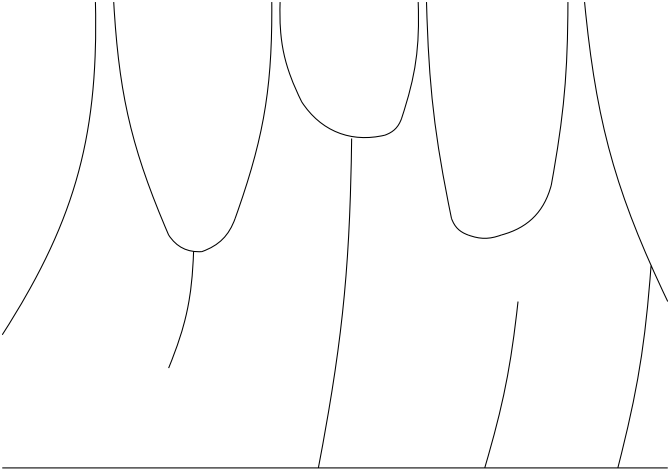

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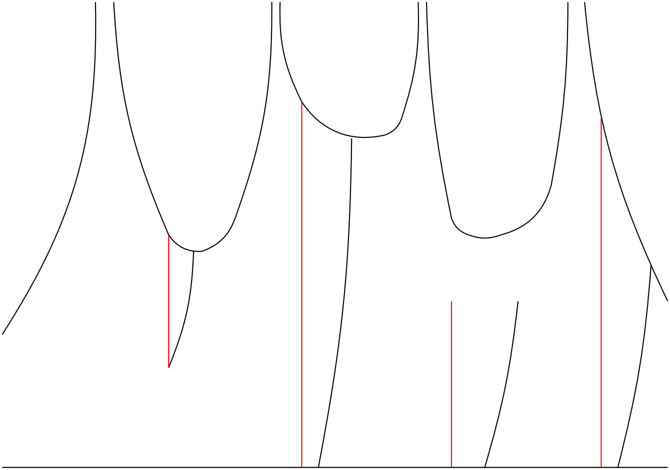

Quantitative shearing

Let $\gamma_{[a,b)}(s) = (s, 0)$ for $a \leq s < b$ be a horizontal segment. Since T is a piecewise isometry, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi_t \circ \gamma(s) = \frac{\mathrm{d}}{\mathrm{d}s}\left(T^{n(x,t)}s, t - S_{n(x,t)}(f)(s)\right) = (1, -S_{n(x,t)}(f')(s)).$$

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Theorem (Ulcigrai, '07; R. '16)

Suppose $C^+ > C^-$. For each $M > 1$, there exist $t_0 > 0$, constants $C, C', \widetilde{C}', C'' > 0$ and $0 < \alpha < 1$ such that for all $t \geq t_0$ there exists $Y(t) \subset I$ with measure greater than $1 - C/(\log t)^\alpha$ such that for all $x \in Y(t)$

- $S_{n(x,t)}(f')(x) \leq -C' t \log t$,
- $\left|S_{n(x,t)}(f')(x)\right| \leq \widetilde{C}' t \log t$,
- $S_{n(x,t)}(f'')(x) \leq (C''/M) t^2 (\log t)^{1+\alpha}$.

Birkhoff sums of f'

A key tool to estimate $S_n(f')$ is the Rauzy-Veech induction.
RV produces a sequence of IETs $(T^{(n)})_n$ on intervals $I^{(n)}$ shrinking to 0. Let $\underline{\lambda}^{(n)}$ be the length vector associated to $T^{(n)}$ and $\underline{h}^{(n)}$ the vector of return times. Then

$$\underline{\lambda}^{(n)} = (A^{(n)})^{-1}\underline{\lambda} \quad \text{and} \quad \underline{h}^{(n)} = (A^{(n)})^T\underline{1},$$

where $(A^{(n)})^{-1} \in \mathrm{SL}_d(\mathbb{Z})$ is the Rauzy-Veech length cocycle.

Birkhoff sums of f'

The full-mesure set of IETs we consider contains those for which there exists a sequence $(n_l)_l$ of induction times such that the return times and the lengths are balanced, namely there exists a constant $\kappa > 1$ such that

$$\kappa^{-1} \leq \frac{\lambda_i^{(n_l)}}{\lambda_j^{(n_l)}}, \frac{h_i^{(n_l)}}{\lambda_j^{(n_l)}} \leq \kappa$$

and there exists $1 < \tau < 2$ for which

$$\lim_{l\to\infty} \frac{\|A^{(n_l,n_{l+1})}\|}{l^\tau} = 0.$$

Decay of correlations

The set $Y(t)$ where the estimates on $S_{n(x,t)}(f')$ hold is partitioned into intervals J of length of order $1/(t \log t)^{\alpha}$.

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Let $\mathcal{X}(t)$ be the set consisting of full vertical translates of $J \subset Y(t)$, namely

$$\mathcal{X}(t) = \bigcup_{J\subset Y(t)} \{(x,y) : x \in J, 0 \le y \le \inf_{x\in J} f(x)\}.$$

We can bound its measure

$$\mathrm{Leb}(\mathcal{X} \setminus \mathcal{X}(t)) = O\Big(\frac{1}{(\log t)^\beta}\Big),$$

for some $0 < \beta < 1$.

Decay of correlations

Let $g, h \in \mathscr{C}_c^1(\mathcal{X})$, with $\int_{\mathcal{X}} g \ \mathrm{d}\,\mathrm{Leb} = 0$. We estimate the decay of correlations

$$\int_{\mathcal{X}} (g \circ \phi_t) h \ \mathrm{d}\,\mathrm{Leb} = \int_{\mathcal{X}(t)} (g \circ \phi_t) h \ \mathrm{d}\,\mathrm{Leb} + O\Big(\frac{1}{(\log t)^{\beta}} \Big).$$

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By Fubini and integration by parts, the modulus of the first term in the RHS above is bounded by

$$\text{const} \sum_{J \in X(t)} \int_0^{\inf_J f} \sup_{a \le u \le b} \left| \int_a^u g \circ \phi_{t+y} \circ \gamma_J(s) \, \mathrm{d}s \right| \mathrm{d}y,$$

Bootstrap trick

We use a bootstrap trick inspired by Forni-Ulcigrai, '12. Fix $\varepsilon > 0$.

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$$+ \int_a^u (g \circ \phi_t \circ \gamma_J(s))\Big(1 + \frac{S_{n(s,t)}(f')(s)}{(C'+\varepsilon)t\log t}\Big)\,\mathrm{d}s.$$

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The first term in the RHS is by definition

$$\frac{1}{(C' + \varepsilon)t \log t} \int_{\phi_t \circ \gamma_{[a,u)}} g\,\mathrm{d}y.$$

Bootstrap trick

Thus

$$\int_a^u g \circ \phi_t \circ \gamma_J(s)\,\mathrm{d}s = \frac{1}{(C' + \varepsilon)t \log t} \int_{\phi_t \circ \gamma_{[a,u)}} g\,\mathrm{d}y$$

$$+ \int_a^u (g \circ \phi_t \circ \gamma_J(s)) \Big(1 + \frac{S_{n(s,t)}(f')(s)}{(C' + \varepsilon)t \log t} \Big)\,\mathrm{d}s$$

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$$\int_a^u g \circ \phi_t \circ \gamma_J(s)\,\mathrm{d}s = \frac{1}{(C' + \varepsilon)t\log t}\int_{\phi_t \circ \gamma_{[a,u)}} g\,\mathrm{d}y$$
$$+ \int_a^u (g \circ \phi_t \circ \gamma_J(s))\Big(1 + \frac{S_{n(s,t)}(f')(s)}{(C' + \varepsilon)t\log t}\Big)\,\mathrm{d}s$$

We integrate the second term by parts and get

$$\Big(1 + \frac{S_{n(s,t)}(f')(s)}{(C' + \varepsilon)t\log t}\Big)\int_a^u g \circ \phi_t \circ \gamma_J(s)\,\mathrm{d}s$$
$$- \int_a^u \Big(\frac{S_{n(s,t)}(f'')(s)}{(C' + \varepsilon)t\log t}\Big)\Big(\int_a^\ell g \circ \phi_t \circ \gamma_J(\ell)\,\mathrm{d}\ell\Big)\,\mathrm{d}s$$

Bootstrap trick

$$\left|\int_a^u g\circ\phi_t\circ\gamma_J(s)\,\mathrm{d}s\right| \le \frac{1}{(C'+\varepsilon)t\log t}\left|\int_{\phi_t\circ\gamma_{[a,u)}} g\,\mathrm{d}y\right| + \left(1-\frac{C'}{C'+\varepsilon}+\frac{C''}{(C'+\varepsilon)M}\right)\sup_{a\le\ell\le u}\int_a^\ell (g\circ\phi_t\circ\gamma_J(s))\,\mathrm{d}s$$

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Choosing $M>1$ so that $C^{-1} := C' - C''/M > 0$ and taking the supremum on both sides,

$$\sup_{a\leq u\leq b}\left|\int_a^u g\circ\phi_t\circ\gamma_J(s)\,\mathrm{d}s\right| \leq \frac{C}{t\log t}\left|\int_{\phi_t\circ\gamma_{[a,u)}} g\,\mathrm{d}y\right|$$

Deviations of ergodic averages for IETs

Using Green's Theorem,

$$\left|\int_{\phi_t\circ\gamma_{[a,u)}} g\,\mathrm{d}y\right| \leq \left|\int_{\xi} g\,\mathrm{d}y\right| + O((\log t)^{-1}),$$

for an appropriate orbit segment $\xi(s) = \phi_{t+s}(a,0)$ of length $O((\log t)^{1-\alpha})$.

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Theorem (Athreya-Forni, '08)

For every permutation π of d elements there exists $0 \leq \theta < 1$ such that for almost every IET $T = T(\pi, \underline{\lambda})$, for every function h in a certain class, there exists $C_h > 0$ for which

$$\left|S_r(h)(x) - r\int_0^1 h(x)\,\mathrm{d}x\right| \leq C_h r^{\theta},$$

uniformly on $x \in I$.

Final estimates

We obtained

$$\left|\sup_{a\leq u\leq b}\int_a^u g\circ\phi_t\circ\gamma_J(s)\,\mathrm{d}s\right| = O\Big(\frac{(\log t)^{\theta(1-\alpha)}}{t\log t}\Big).$$

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We conclude

$$\left| \int_{\mathcal{X}} (g \circ \phi_t) h \;\mathrm{d}\,\mathrm{Leb} \right| \le O\Big(\frac{(\log t)^{\theta(1-\alpha)}}{t \log t} \Big) \sum_{J \in X(t)} \int_0^{\inf_J f} \mathrm{d}y$$

$$\le O\Big(\frac{(\log t)^{\theta(1-\alpha)}}{t \log t} \cdot \frac{1}{\mathrm{Leb}(J)} \Big) = O\Big(\frac{1}{(\log t)^{(1-\theta)(1-\alpha)}} \Big)$$

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