# A geometric approach for constructing equilibrium measures in hyperbolic dynamics

Yakov Pesin
Pennsylvania State University
Joint work with V. Climenhaga and A. Zelerowicz

DinAmicI V - Modern Trends in the Ergodic Theory of Dynamical Systems

May 31, 2017

# Equilibrium Measures

$X$ a compact metric space with metric $d$;
$f : X \to X$ a continuous map of finite topological entropy;
$\varphi$ a continuous function (potential) on $X$;
$\mathcal{M}(f, X)$ the space of all $f$-invariant Borel probability measures on $X$;
$\mu_\varphi \in \mathcal{M}(f, X)$ is an equilibrium measure if

$$P(\varphi) := -\inf_{\mu \in \mathcal{M}(f,X)} E_\mu(\varphi) = -E_{\mu_\varphi}(\varphi),$$

where $P(\varphi)$ is the topological pressure of $\varphi$ and

$$E_\mu(\varphi) = -(h_\mu(f) + \int_X \varphi \, d\mu)$$

the free energy of the system w.r.t. $\mu$. It suffices to take the infimum over the space $\mathcal{M}^e(f, X) \subset \mathcal{M}(f, X)$ of ergodic measures.

# Equilibrium Measures for Uniformly Hyperbolic Maps (Sinai, Ruelle, Bowen)

$X$ a compact smooth manifold of dimension $\geq 2$;
$f : X \to X$ a diffeomorphism;
$\Lambda \subset X$ a locally maximal hyperbolic set for $f$; the hyperbolic structure is generated by a $df$-invariant splitting of the tangent space: for every $x \in \Lambda$ we have $T_x M = E^s(x) \oplus E^u(x)$ with uniform contraction along stable $E^s(x)$ and uniform expansion along unstable $E^u(x)$ subspaces; we assume that $f|\Lambda$ is topologically transitive.

**Theorem**

*Assume $\varphi$ is Hölder continuous. Then*

- *Existence: there is an equilibrium measure $\mu_\varphi$;*
- *Uniqueness: $\mu_\varphi$ is a unique equilibrium measure;*
- *Ergodic properties: $\mu_\varphi$ is ergodic and in fact is Bernoulli (up to a rotation);*

# The Geometric Potential

Of special interest is the geometric $t$-potential: a family of potential functions $\varphi_t(x) = -t \log |\text{Jac}(df|E^u(x))|$ for $t \in \mathbb{R}$. Since the subspaces $E^u(x)$ depend Hölder continuously on $x$ for each $t$ the potential $\varphi_t$ is Hölder continuous and hence, admits a unique equilibrium measure $\mu_t$.

# The Geometric Potential

Of special interest is the geometric $t$-potential: a family of potential functions $\varphi_t(x) = -t \log |\text{Jac}(df|E^u(x))|$ for $t \in \mathbb{R}$. Since the subspaces $E^u(x)$ depend Hölder continuously on $x$ for each $t$ the potential $\varphi_t$ is Hölder continuous and hence, admits a unique equilibrium measure $\mu_t$.

- The pressure function $P(t) := P(\varphi_t)$ is convex, decreasing and real analytic in $t$.
- $P(0) = h_{\text{top}}(f)$ is the topological entropy of $f$ and $\mu_0$ is the unique measure of maximal entropy;
- there is unique $0 < t_0 \leq 1$ for which $P(t_0) = 0$ is the root of Bowen's equation; in the two-dimensional case the equilibrium measure $\mu_{t_0}$ is the measure of maximal Hausdorff dimension of the set $\Lambda \cap V^u(x)$ where $V^u(x)$ is a local unstable leaf through $x$ (this is independent of $x \in \Lambda$); the result extends to the multi-dimensional case provided the action of $df$ on $E^u(x)$ is conformal; otherwise, the measure of maximal Hausdorff dimension may not exist.

- Consider the particular case when $\Lambda$ is a topological attractor for $f$ that is there is a neighborhood $U$ of $\Lambda$ such that $\overline{f(U)} \subset U$; this implies that

$$\Lambda = \bigcap_{n \geq 0} f^n(U);$$

in this case $t_0 = 1$ and $\mu_1$ is the unique Sinai-Ruelle-Bowen (SRB) measure for $f$: the conditional measures generated by $\mu_1$ on unstable local leaves $V^u(x)$ are equivalent to the leaf volume $m^u(x)$ on $V^u(x)$.

# The Symbolic Approach for Constructing Equilibrium Measures

The proof uses a Markov partition $\mathcal{P} = \{P_1, \dots, P_\ell\}$ of $\Lambda$ to construct a symbolic representation of $f|\Lambda$ by a subshift of finite type $(\Sigma_A, \sigma)$ where

- the transition matrix $A = (a_{ij})$ is given by $a_{ij} = 1$ if $P_i \cap f^{-1}(P_j) \neq \emptyset$ and $a_{ij} = 0$ otherwise;
- $\Sigma_A$ is the set of two-sided infinite sequences $x = (x_i)$ which are admissible by $A$, i.e., $a_{x_i x_{i+1}} = 1$;
- $\sigma$ is the full shift.

One then uses a Ruelle's version of the classical Perron–Frobenius theorem for matrices to establish existence of equilibrium measures:

- Observe that $\Sigma_A$ is a natural extension of the space $\Sigma_A^+$ of one-sided infinite sequences admissible by $A$.
- Consider the Ruelle operator on the space $C(\Sigma_A^+)$

$$(\mathcal{L}_\varphi f)(x) = \sum_{\sigma(y)=x} e^{\varphi(y)} f(y).$$

- There exist $\lambda > 0$, a continuous positive function $h$, and a Borel measure $\nu$ such that $\mathcal{L}_\varphi h = \lambda h$ and $\int_{\Sigma_A^+} h\, d\nu = 1$ (i.e., $h$ is a normalized eigenfunction for the Ruelle operator);
- The measure $\mu_\varphi = h\, d\nu$ is a $\sigma$-invariant Gibbs measure for $\varphi$, i.e., there are $C_1 > 0$, $C_2 > 0$ s.t. for every $n > 0$ and $x = (x_i) \in \Sigma_A^+$,

$$C_1 \le \frac{\mu_\varphi(C_n(x))}{\exp\left(-P(\varphi)n + \sum_{k=0}^{n-1} \varphi(\sigma^k(x))\right)} \le C_2,$$

where $C_n(x) = \{y = (y_i) \colon y_i = x_i, i = 0, \dots, n\}$ is a cylinder.
- Every Gibbs measure is an equilibrium measure for $\varphi$.

# The Geometric Approach for Constructing Equilibrium Measures

The idea of the geometric approach is to follow the classical Bogolyubov-Krylov procedure for constructing invariant measures by pushing forward a given reference measure. To this end fix $x \in \Lambda$ and let $\nu^u(x)$ be a Borel probability measure on $V^u(x) \cap \Lambda$. We can extend this measure to a measure on the whole $\Lambda$ (still denoted by $\nu^u(x)$). Consider the sequence of probability measures on $\Lambda$

$$\mu_n^u(x) = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \nu^u(x).$$

Any weak* limit of this sequence is an invariant measure on $\Lambda$. The sequence of measures $\mu_n^u(x)$ describe the evolution of the reference measure $\nu^u(x)$ under the dynamics.

To see that this procedure can be used to obtain equilibrium measures assume that $\mu_\varphi$ is an equilibrium measure for a potential $\varphi$. This measure generates the system of conditional measures $\mu^u(x)$ on unstable local leaves $V^u(x)$.

**Theorem**

*For almost every $x \in \Lambda$, setting the reference measure $\nu^u(x)$ to be the conditional measure $\mu^u(x)$, the sequence of measures $\mu_n^u(x)$ converges to $\mu$.*

Consider the case when $\Lambda$ is a hyperbolic attractor for $f$. Note that $V^u(x) \subset \Lambda$ for every $x \in \Lambda$. In this case the geometric approach can be used to construct the SRB measure on $\Lambda$ which is the equilibrium measure for the potential $\varphi_1$.

# The Geometric Approach for Constructing SRB Measures

Consider the case when $\Lambda$ is a hyperbolic attractor for $f$. Note that $V^u(x) \subset \Lambda$ for every $x \in \Lambda$. In this case the geometric approach can be used to construct the SRB measure on $\Lambda$ which is the equilibrium measure for the potential $\varphi_1$.
Let us choose a local unstable manifold $V^u(x_0)$.

**Theorem**

*Setting the reference measure $\nu^u(x_0)$ to be the leaf volume $m^u(x_))$ on $V^u(x_0)$, the sequence of measures $\mu_n^u(x_0)$ converges to an SRB measure on $\Lambda$.*

# Outline of the proof

A local unstable manifold $V^u(x)$, $x \in \Lambda$ can be expressed in the following way: $V^u(x) = \exp_x\{(w, \psi(w))\}$ where $\psi \in C^{1+\alpha}(B^u[0,r], E^s(x))$. Write $\mathbf{I} = (\gamma, \kappa, r)$ and consider the space of local unstable manifolds $\mathcal{R}_{\mathbf{I}}$ for which the corresponding functions $\psi$ satisfy

$$\|d\psi\| \le \gamma, \quad |d\psi|_\alpha \le \kappa.$$

Given $W \in \mathcal{R}_{\mathbf{I}}$, consider a standard pair $(W, \rho)$ where $\rho$ is a continuous "density" function on $W$ (Chernov and Dolgopyat). Fix $L > 0$, write $\mathbf{K} = (\mathbf{I}, L)$, and consider the space of standard pairs

$$\mathcal{R}'_{\mathbf{I}} = \{(W, \rho) : \rho \in C^\alpha(W, [L^{-1}, L]), |\rho|_\alpha \le L\}.$$

The spaces $\mathcal{R}_{\mathbf{I}}$ and $\mathcal{R}'_{\mathbf{I}}$ have natural local product coordinate systems and are compact in the product topology.

A measure $\eta$ on $\mathcal{R}'_{\mathbf{l}}$ determines a measure $\Phi(\eta)$ on $\Lambda$ by

$$\Phi(\eta)(E) = \int_{\mathcal{R}'_{\mathbf{l}}} \int_{E\cap W} \rho \, dm_W \, d\eta(W,\rho), \quad E \subset \Lambda.$$

Write $\mathcal{M}(\Lambda)$ and $\mathcal{M}(\mathcal{R}'_{\mathbf{l}})$ for the spaces of finite Borel measures on $\Lambda$ and $\mathcal{R}'_{\mathbf{l}}$, respectively. One can show that $\Phi\colon \mathcal{M}(\mathcal{R}'_{\mathbf{l}}) \to \mathcal{M}(\Lambda)$ is continuous; in particular, $\mathcal{M}_{\mathbf{K}} = \Phi(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{l}}))$ is compact, where $\mathcal{M}_{\leq 1}$ is the space of measures with total weight at most 1.

A measure $\eta$ on $\mathcal{R}'_{\mathbf{l}}$ determines a measure $\Phi(\eta)$ on $\Lambda$ by

$$\Phi(\eta)(E) = \int_{\mathcal{R}'_{\mathbf{l}}} \int_{E \cap W} \rho \, dm_W \, d\eta(W, \rho), \quad E \subset \Lambda.$$

Write $\mathcal{M}(\Lambda)$ and $\mathcal{M}(\mathcal{R}'_{\mathbf{l}})$ for the spaces of finite Borel measures on $\Lambda$ and $\mathcal{R}'_{\mathbf{l}}$, respectively. One can show that $\Phi \colon \mathcal{M}(\mathcal{R}'_{\mathbf{l}}) \to \mathcal{M}(\Lambda)$ is continuous; in particular, $\mathcal{M}_{\mathbf{K}} = \Phi(\mathcal{M}_{\leq 1}(\mathcal{R}'_{\mathbf{l}}))$ is compact, where $\mathcal{M}_{\leq 1}$ is the space of measures with total weight at most 1.
An invariant probability measure is an SRB measure if and only if it is in $\mathcal{M}_{\mathbf{K}}$ for some $\mathbf{K}$.
One can show that $\mathcal{M}_{\mathbf{K}}$ is invariant under the action of $f_*$, and thus $\mu_n^u(x_0) \in \mathcal{M}_{\mathbf{K}}$ for every $n$. By compactness of $\mathcal{M}_{\mathbf{K}}$, one can pass to a subsequence $\mu_{n_k}$ which converges to a measure $\mu \in \mathcal{M}_{\mathbf{K}}$, and this is the desired SRB measure.

# Other Non-symbolic Approaches for Constructing Equilibrium Measures

**1.** Margulis' approach for constructing measures of maximal entropy for Anosov systems: the iteration procedure is realized by the action of a linear operator generated by the map $f$ on the Banach space of linear functionals on the space of leaf-wise continuous functions with leaf-wise compact support. The proof uses in a substantial way the holonomy map generated by global stable leaves. Geometrically this approach corresponds to the inverse iteration procedure.

# Other Non-symbolic Approaches for Constructing Equilibrium Measures

**1.** Margulis' approach for constructing measures of maximal entropy for Anosov systems: the iteration procedure is realized by the action of a linear operator generated by the map $f$ on the Banach space of linear functionals on the space of leaf-wise continuous functions with leaf-wise compact support. The proof uses in a substantial way the holonomy map generated by global stable leaves. Geometrically this approach corresponds to the inverse iteration procedure.

**2.** Approach by Gouezel and Liverani – the push forward of the leaf-volume $m^u(x)$ by the Ruelle operator: for a standard pair $(W,\rho)$ we have $\mathcal{R}(W,\rho) = (f(W), R_W\rho)$ where $(R_W\rho)(f(x)) = e^{\varphi(x)}\rho(x)$. For $\varphi = \varphi_1$ this is the standard push forward of the leaf volume; in general a limit measure may not be invariant.

Let $X$ be a set and $\mathcal{F}$ a collection of subsets of $X$ called admissible. Assume that there are two set functions $\eta, \psi : \mathcal{F} \to [0, \infty)$ satisfying

(A1) $\emptyset \in \mathcal{F}$; $\eta(\emptyset) = \psi(\emptyset) = 0$ and $\eta(U), \psi(U) > 0$ for any $U \in \mathcal{F}, U \neq \emptyset$;

(A2) for any $\delta > 0$ one can find $\varepsilon > 0$ such that $\eta(U) \leq \delta$ for any $U \in \mathcal{F}$ with $\psi(U) \leq \varepsilon$;

(A3) there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, one can find a finite or countable subcollection $\mathcal{G} \subset \mathcal{F}$ covering $X$ such that $\psi(U) \leq \varepsilon$ for any $U \in \mathcal{G}$.

Let $\xi : \mathcal{F} \to [0, \infty)$ be a set function. The collection of subsets $\mathcal{F}$ and the functions $\xi, \eta, \psi$, satisfying (A1), (A2), (A3) introduce a Carathéodory dimension structure or $C$-structure $\tau = (\mathcal{F}, \xi, \eta, \psi)$ on $X$.

$\eta$ is a potential set function, $\xi$ measures the weight, and $\psi$ the size of $U \subset \mathcal{F}$.

For any subcollection $\mathcal{G} \subset \mathcal{F}$ let $\psi(\mathcal{G}) := \sup\{\psi(U) : U \in \mathcal{G}\}$. Given $Z \subset X$ and numbers $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, define

$$M_C(Z, \alpha, \varepsilon) := \inf_{\mathcal{G}, \psi(\mathcal{G}) \leq \varepsilon} \left\{ \sum_{U \in \mathcal{G}} \xi(U) \eta(U)^\alpha \right\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{F}$ covering $Z$. Set

$$m_C(Z, \alpha) := \lim_{\varepsilon \to 0} M_C(Z, \alpha, \varepsilon).$$

If $m_C(\emptyset, \alpha) = 0$, the set function $m_C(\cdot, \alpha)$ becomes an outer measure on $X$, which induces a measure called the $\alpha$-Carathéodory measure. In general, this measure may not be $\sigma$-finite or it may be a zero measure.

Furthermore, there exists $\alpha_C \in \mathbb{R}$ s.t. $m_C(Z, \alpha) = \infty$ for $\alpha < \alpha_C$ and $m_C(Z, \alpha) = 0$ for $\alpha > \alpha_C$ (while $m_C(Z, \alpha_C)$ may be 0, $\infty$, or a finite positive number). The quantity $\dim_C Z = \alpha_C$ is the Carathéodory dimension of $Z$.

# The Hausdorff Dimension Structure

A $C$-structure can be generated by other structures on the set $X$. For example, if $X$ is a metric space, then consider the $C$-structure given by

$$\mathcal{F} := \{\text{open sets}\}, \quad \xi(U) = 1, \quad \eta(U) = \psi(U) = \text{diam } U.$$

For any $Z \subset X$ we have that $\dim_C Z = \dim_H Z$ is the Hausdorff dimension of $Z$.
Other examples of Carathéodory dimensions are dimension spectra, dimension of Poincaré recurrences, topological entropy and topological pressure.

# Carathéodory Structure on Local Unstable Manifolds

Fix $x_0 \in \Lambda$ and set $X := V^u(x_0) \cap \Lambda$. Fix a small number $r > 0$ and define the Bowen's $u$-ball by

$$B_n^u(x,r) := \{y \in V^u(x) \cap \Lambda : d(f^k(y), f^k(x)) < r \text{ for } k = 0, \dots, n\}.$$

Then define the collection $\mathcal{F}$ of admissible sets by

$$\mathcal{F} := \{\emptyset\} \cup \{B_n^u(x,r) : x \in V^u(x_0) \cap \Lambda, n \in \mathbb{N}\}.$$

Given $x \in X$ and $n > 0$, define

$$S_n\varphi(x) := \exp\left(\sum_{k=0}^{n-1} \varphi(f^k(x))\right),$$

$$\xi(B_n^u(x,r)) := \exp(S_n\varphi(x)),\ \eta(B_n^u(x,r)) := e^{-n},\ \psi(B_n^u(x,r)) := \frac{1}{n},$$

and also set $\eta(\emptyset) = \psi(\emptyset) = 0$ and $\xi(\emptyset) = 1$. It is easy to see that the collection of subsets $\mathcal{F}$ and set functions $\xi, \eta, \psi$ satisfy (A1), (A2), (A3), and hence, introduce a $C$-structure in $X$.

Thus we obtain the Carathéodory measure $m^u_{C,x_0} := m_{C,x_0}(\cdot, P)$ on $X$ at the Carathéodory dimension $P = \dim_C X$. For every $Z \subset X$ we have

$$m^u_{C,x_0}(Z) = \lim_{N\to\infty} \inf_{\{B^u_{n_i}(x_i,r)\}} \left\{ \sum_i \exp\left( -Pn_i + \sum_{k=0}^{n_i-1} \varphi(f^k(x_i)) \right) \right\},$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, r)\}$ of $u$-Bowen's balls with $x_i \in X$, $n_i \geq N$, which cover $Z$ that is $Z \subset \bigcup_i B^u_{n_i}(x_i, r)$.

The fact that the the Carathéodory structure and hence, the Carathéodory measure do not depend on $r$ is due to expansivity of the map $f|\Lambda$: there is $\delta > 0$ such that no two trajectories can stay within the distance $\delta$ from each other.

# Main Results

**Theorem (1)**

*The P-Carathéodory measure $m^u_{C,x_0}$ on $X = V^u(x_0) \cap \Lambda$ is finite and positive independently of the choice of the number $r$ provided it is sufficiently small. Moreover, $P = P(\varphi)$.*

As an immediate corollary we obtain that for any set $Z \subset X$ of positive $m^u_{C,x_0}$-measure we have that $\dim_C Z = P$.

# Main Results

**Theorem (1)**

*The P-Carathéodory measure $m^u_{C,x_0}$ on $X = V^u(x_0) \cap \Lambda$ is finite and positive independently of the choice of the number $r$ provided it is sufficiently small. Moreover, $P = P(\varphi)$.*

As an immediate corollary we obtain that for any set $Z \subset X$ of positive $m^u_{C,x_0}$-measure we have that $\dim_C Z = P$.

**Theorem (2)**

*Let $\Lambda$ be a locally maximal hyperbolic set for a diffeomorphism $f$. Assume that $f|\Lambda$ is topologically transitive. Then for any $x_0 \in \Lambda$ setting the reference measure $\nu^u(x_0)$ to be the P-Carathéodory measure $m^u_{C,x_0}$ on $V^u(x_0)$, the sequence of measures $\mu_n(x_0)$ converges in the weak$^*$ topology to a measure $\mu_\varphi$ which is the unique equilibrium measure for $\varphi$.*

# Proof of Theorem (1)

The proof is based on the following facts:

- Expansivity of $f$;
- Uniform contraction and expansion along local stable and unstable manifolds;
- Bowen's property of the potential: there exist $K > 0$ and $r_0 > 0$ such that for every $x \in \Lambda$ and $n > 0$ if $y \in B_n(x, r_0)$ (Bowen's ball at $x$ of radius $r$), then $|S_n\varphi(x) - S_n\varphi(y)| \leq K$;

Using this properties one can prove that for $Y \subset \Lambda$, $n > 0$, and $\delta > 0$,

$$C^{-1}e^{nP} \leq N(Y, n, \delta) \leq Ce^{nP},$$

where $C > 0$ is a constant,

$$N(Y, \delta, n) := \sup \left\{ \sum_{x \in E} e^{S_n\varphi(x)} \right\}$$

and the supremum is taken over all $(\delta, n)$-separated subsets $E \subset Y$.

To show that the measure $m^u_{C,x_0}$ on $X = V^u(x_0) \cap \Lambda$ is finite it suffices to take a finite cover of $X$ by Bowen's $u$-balls $B^u_n(x_i, r)$ centered at points $x_i$, $i = 1, \dots, \ell$ that form a maximal $(n, \frac{r}{2})$-separated set. Then

$$\begin{aligned}
&\sum_{i=1}^{\ell} \exp\left(-Pn_i + S_n\varphi(y_i)\right) : y_i \in X, \\
&\leq e^{K-Pn} \sum_{i=1}^{\ell} e^{S_n\varphi(x_i)} \\
&\leq Ce^{K-Pn} e^{nP} \leq Ce^K,
\end{aligned}$$

where $C > 0$ is a constant.

To show that the measure $m^u_{C,x_0}$ on $X = V^u(x_0) \cap \Lambda$ is positive consider any cover of $X$ by Bowen's $u$-balls $B^u_{n_i}(x_i, r)$, $i = 1, 2, \dots$ and choose a finite subcover $B^u_{n_i}(x_i, r)$ with $n_i < M$. One can then show that

$$\sum_i \exp(-Pn_i + S_{n_i}\varphi(y_i)) \geq C_1 \frac{N(\mathcal{O}_r, r, M)}{N(\Lambda, r, M)},$$

where $C_1 > 0$ is a constant and $\mathcal{O}_r(V)$ is a neighborhood of $V$ in $\Lambda$ of size $r$. Then

$$\frac{N(\mathcal{O}_r, r, M)}{N(\Lambda, r, M)} \geq \frac{C_2}{N(\Lambda, r, M)} \sum_{x \in E} e^{S_M \varphi(x)} \geq C_3,$$

where $C_2 > 0$ and $C_3 > 0$ are constants and $E$ is a maximal $(M, \frac{r}{2})$-separated set.

# Proof of Theorem (2)

The proof follows the line of argument in the construction of the SRB measure described above. As a result one gets that a limit measure has conditional measures on unstable leaves which are equivalent to measures $m^u_{C,x}$. One then show that $\mu$ has the $u$-Gibbs property: there are $C_1 > 0$, $C_2 > 0$ such that for every $n > 0$, $x \in V^u(x) \cap \Lambda$, and any sufficiently small $r > 0$,

$$C_1 \le \frac{\mu_\varphi(B^u_n(x,r))}{\exp(-Pn + S_n\varphi(x))} \le C_2.$$

One deduces from here that $\mu$ has the Gibbs property: there are $C_3 > 0$, $C_4 > 0$ such that for every $n > 0$, $x \in \Lambda$, and any sufficiently small $r > 0$,

$$C_3 \le \frac{\mu_\varphi(B_n(x,r))}{\exp(-Pn + S_n\varphi(x))} \le C_4.$$

This immediately implies that $\mu_\varphi$ is an equilibrium measure.

# Comments

**1.** The theorems extend to the class of potentials $\varphi$ satisfying Bowen's property, which is larger than the class of Hölder continuous potentials.

**2.** Consider the geometric $t$-potential $\varphi_t$ on $\Lambda$ and let $t_0$ be the unique root of Bowen's equation $P(t_0) = 0$. There is a $C$-structure on $X = V^u(x_0) \cap \Lambda$ for which $t_0$ is the Carathéodory dimension of the set $X$. Indeed, since $P(t_0) = 0$, we have

$$m^u_{C,x_0}(Z) = \lim_{N\to\infty} \inf \left\{ \sum_{\{B^u_{n_i}(x_i,r)\}} \left( \prod_{k=0}^{n_i-1} \mathrm{Jac}\,(df|E^u(f^k(x_i)) \right)^{t_0} \right\}.$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, r)\}$ of Bowen's $u$-balls with $x_i \in X$, $n_i \geq N$, which cover $Z$. Hence, $m^u_{C,x_0}$ can be viewed as the Carathéodory measure generated by the $C$-structure $\tau' = (\mathcal{F}, \xi', \eta', \psi)$, where $\xi'(B^u_n(x, r)) := 1$ and

$$\eta'(B^u_n(x, r)) := \prod_{k=0}^{n_i-1} \mathrm{Jac}\,(df|E^u(f^k(x_i)).$$

We have $\dim_{C,\tau'} X = t_0$ and $m^u_{C,x_0} = m_{C,\tau'}(\cdot, t_0)$ is the measure of maximal Carathéodory dimension. If $f$ is conformal in the unstable direction, then $m^u$ is the measure of full Hausdorff dimension.

**3.** The main theorems extend to some partially hyperbolic systems such as the time-1 map of an Anosov flow and its small perturbations and the time-1 map of the frame flow on any manifold of generic negative curvature (Spatzier and Visscher) and its small perturbations.