Modern Trends in the Ergodic Theory of Dynamical Systems
DinAmici V, INdAM, Rome

Diophantine type and translation flows

Stefano Marmi
joint work with Dong Han Kim and Luca Marchese

Scuola Normale Superiore, Pisa

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Hitting time and return time

$\phi : \mathbb{R} \times X \to X$, $(t, p) \mapsto \phi^t(p)$ minimal, X compact metric space. Let $p, p' \in X$, $r > 0$ small enough. The *hitting time* of p to the ball $B(p', r)$ of radius r around p' is

$$R(\phi, p, p', r) := \inf\{t > r \mid \mathrm{Dist}(\phi^t(p), p') < r\}$$

The same quantity is also obviously defined for discrete time dynamical systems $t \in \mathbb{Z}$. Scaling exponents when $r \to 0$:

$$\underline{w}_{\mathrm{hit}}(\phi, p, p') := \liminf_{r \to 0^+} \frac{\log R(\phi, p, p', r)}{-\log r}$$

$$\overline{w}_{\mathrm{hit}}(\phi, p, p') := \limsup_{r \to 0^+} \frac{\log R(\phi, p, p', r)}{-\log r}.$$

The *return time* is obtained taking $p' = p$: $R(\phi, p, r) := R(\phi, p, p, r)$ and analogously

$$\underline{w}_{\mathrm{ret}}(\phi, p) := \underline{w}_{\mathrm{hit}}(\phi, p, p) \quad \text{and} \quad \overline{w}_{\mathrm{ret}}(\phi, p) := \overline{w}_{\mathrm{hit}}(\phi, p, p).$$

Circle rotations: diophantine type and scaling exponents

Kim and Seo (2003) proved that for an irrational circle rotation $T_\alpha : \mathbb{T} \to \mathbb{T}$, $T_\alpha(x) = x + \alpha$, for a.e. $x, x' \in \mathbb{T}$ one has

$$\underline{w}_{\mathrm{hit}}(T_\alpha, x, x') = 1 \quad \text{and} \quad \overline{w}_{\mathrm{hit}}(T_\alpha, x, x') = w(\alpha)$$

where $w(\alpha) \geqslant 1$ is the *diophantine type* of α, i.e. the supremum of those $w \geqslant 1$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{w+1}}$ has infinitely many solutions $p/q \in \mathbb{Q}$.

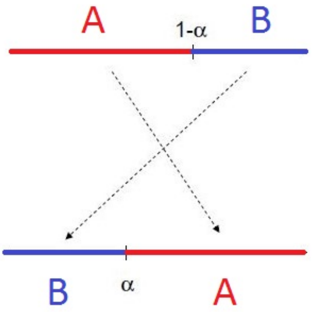
Moreover it is easy to check (Choe and Seo, 2001) that for all $x \in \mathbb{T}$

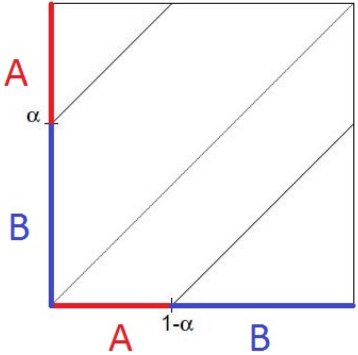
$$\underline{w}_{\mathrm{ret}}(T_\alpha, x) = \frac{1}{w(\alpha)} \quad \text{and} \quad \overline{w}_{\mathrm{ret}}(T_\alpha, x) = 1$$

For almost any α one has $w(\alpha) = 1$ (*Roth type numbers*). Then for any such α and for a.e. $x, x' \in \mathbb{T}$ one has

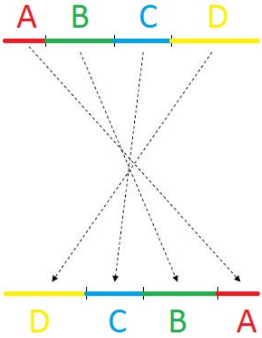
$$\lim_{r \to 0_+} \frac{\log R(T_\alpha, x, r)}{-\log r} = 1 \quad \text{and} \quad \lim_{r \to 0_+} \frac{\log R(T_\alpha, x, x', r)}{-\log r} = 1$$

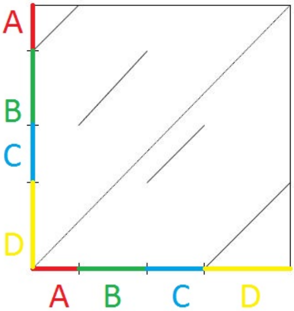
Circle rotations





Standard interval exchange maps





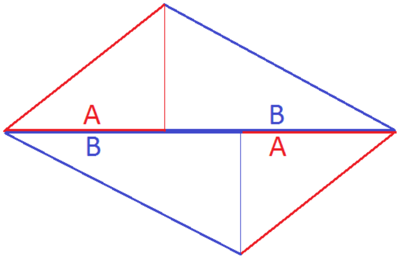
Singularities, connections and minimality

- The *singularities* of T are the $d-1$ points $u_1^t < \cdots < u_{d-1}^t$ separating the subintervals in the domain of T.
- The *singularities* of T^{-1} are the $d-1$ points $u_1^b < \cdots < u_{d-1}^b$ separating the subintervals in the image of T.
- A *connection* is a relation $T^m(u_i^b) = u_j^t$ with $1 \leqslant i, j < d$ and $m \geqslant 0$.

Theorem (Keane) *Let T be a* **standard** *i.e.m.*
If the length data are rationally independent, T has no connection.
If T has no connection, T is minimal: every infinite half-orbit of T is dense.

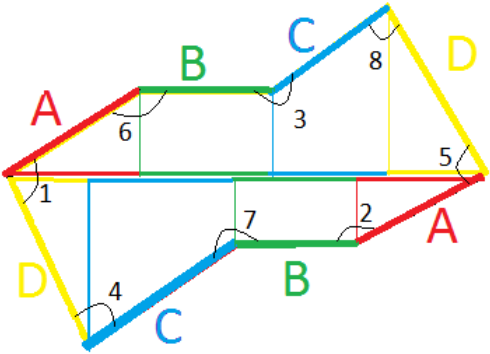
Suspension of i.e.m: from the torus...

Suspending a rotation (after a choice of vertical parameters) produces a flat torus.



..to higher genus: a flat surface of genus 2 with one conical singularity of angle 6π

Suspending an interval exchange map with combinatorics $\pi = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$ produces a translation surface of genus 2.



Translation surfaces

Let $P \subset \mathbb{R}^2$ be a (collection of) polygons in the plane (above $P = \sqcup_{g \in \mathcal{D}} g \cdot Q$) and assume that ∂P is union of $2d \geqslant 4$ segments which come in parallel pairs. A *translation surface* is the quotient space $X = P/\sim$ obtained identifying pairwise parallel sides by a translation. We assume that the identification gives a connected quotient space X.

If P is a parallelogram then $X = P/\sim$ is a flat torus. In general a translation surface is a compact surface of genus $g \geqslant 1$, with a metric which is flat outside of a finite set Σ of points $p_1, \ldots, p_r$ of X, where the metric has a conical singularity with angle $2(k_j + 1)\pi$ and $k_j \in \mathbb{N}$. Any p_j corresponds to a subset of the vertices of P, all identified to the same point in X by the equivalence relation $\sim$ on ∂P. We have

$$k_1 + \cdots + k_r = 2g - 2.$$

Combinatorial invariants of i.e.m.'s

From the combinatorial data of an i.e.m T, one can compute

- the genus g of any surface M obtained by suspension of T;
- the number s of conical singularities of M;
- the angles $2\pi k_i$, $i = 1, \ldots, s$ at these singularities.

The number d of intervals is related to g and s through

$$d = 2g + s - 1.$$

Translation surfaces and billiard flows on rational polygons

Consider the *billiard flow* $\widetilde{\phi}$ in a *rational (connected) polygon* $Q \subset \mathbb{R}^2$ (i.e. whose angles $\in \mathbb{Q} \cdot \pi$). The reflections at its sides generate a finite group $\mathcal{D}$ of affine isometries of $\mathbb{R}^2$. Let D be the finite group made by the linear part of elements of $\mathcal{D}$: any $\theta \in S^1$ has a finite orbit $[\theta]$ and the phase space $Q \times S^1$ is foliated into invariant surfaces $Q \times [\theta]$, all mutually isometric, and the billiard flow acts as a linear flow on each of them. Such invariant surfaces are an example of *translation surfaces*.

Hitting time for interval exchange maps

Kim-M (2008) show that for almost every *interval exchange map* T (any d, g, s) with no connection and for almost every x, x' one has

$$\lim_{r\to 0_+} \frac{\log R(T,x,r)}{-\log r} = 1 \quad \text{and} \quad \lim_{r\to 0_+} \frac{\log R(T,x,x',r)}{-\log r} = 1$$

This result exploits the notion of i.e.m. of Roth type introduced by M-Moussa-Yoccoz in 2005.
On the other hand by Jarník Theorem

$$\dim_H \left(\{\alpha \in \mathbb{R} \mid w(\alpha) = \eta\}\right) = \frac{2}{1+\eta},$$

thus for "many" parameters α one has $\overline{w}_{\rm hit}(T_\alpha, x, x') > 1$ for a.e. x, x'.
The same holds replacing T_α by the linear flow ϕ_θ in direction $\theta = \tan^{-1}\alpha$ on $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$.
Problem: *What is the relation between hitting time and the diophantine exponent in translation flows in higher genus?*

Dynamics in moduli spaces (parameter space dynamics)

A *stratum* $\mathcal{H} = \mathcal{H}(k_1, \ldots, k_r)$ is the set of all translation surfaces X with the same order of conical singularities $k_1, \ldots, k_r$.
$\mathrm{SL}(2,\mathbb{R})$ acts on any stratum $\mathcal{H}$: given $G \in \mathrm{SL}(2,\mathbb{R})$ and any translation surface $X = P/\sim$, we define $G \cdot X$ as the quotient space $G \cdot P/\sim$, where $G \cdot P$ is the affine image of P under the action of G on $\mathbb{R}^2$.

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \; t \in \mathbb{R} \quad \text{and} \qquad r_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;, \theta \in [0, 2\pi) \;.$$

$\mathrm{SL}(2,\mathbb{R})$-orbit of a surface $X \in \mathcal{H}$ is closed in $\mathcal{H}$ if and only if the stabiliser $\mathrm{SL}(X)$ of X under the action of $\mathrm{SL}(2,\mathbb{R})$, called the *Veech group* of X, is a lattice in $\mathrm{SL}(2,\mathbb{R})$. Such a surface is called *Veech surface*
In general the Veech group of any translation surface X is a discrete and non co-compact subgroup of $\mathrm{SL}(2,\mathbb{R})$, and it is trivial for generic X.

Phase space dynamics, holonomy and saddle connections

We denote with ϕ_θ^t the flow at unit speed at time t in direction θ on X.
Let $\gamma \subset X$ be a segment of a geodesic, we denote $\mathrm{Hol}(\gamma, X) \in \mathbb{R}^2$ the planar development of γ in $\mathbb{R}^2$.
For any $G \in \mathrm{SL}(2, \mathbb{R})$ one has $\mathrm{Hol}(\gamma, G \cdot X) = G\big(\mathrm{Hol}(\gamma, X)\big)$.
A *saddle connection* is a geodesic segment γ connecting two conical singularities.
The set $\mathrm{Hol}(X)$ of *relative periods* of X is the set of vectors $\mathrm{Hol}(\gamma) \in \mathbb{R}^2$, where γ is a saddle connection for X.
$\mathrm{Hol}(X) \subset \mathbb{R}^2$ is a discrete subset, but its projectivization is dense in $\mathbb{PR}^2$, thus it is meaningful to consider diophantine approximations of a given direction θ by directions of vectors in $\mathrm{Hol}(X)$.
The components of γ along the direction θ are

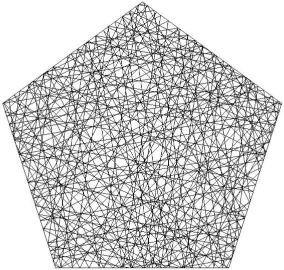
$$\mathrm{Hol}(\gamma, r_{-\theta} \cdot X) = \big(\mathrm{Re}(\gamma, \theta), \mathrm{Im}(\gamma, \theta)\big) \in \mathbb{R}^2.$$

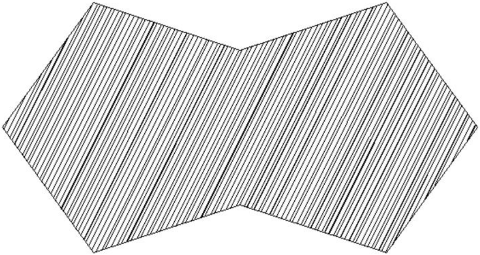
Phase space dynamics and saddle connections

If σ is a *closed geodesic* of X, we denote C_σ the maximal connected open set foliated by closed geodesic parallel to σ (a "cylinder"). By maximality its boundary is union of saddle connections parallel to σ.
Given any direction θ on a translation surface X, a *separatrix* is a trajectory of ϕ_θ starting at a conical point. If $2(k_i + 1)\pi$ is the cone angle at the conical point p_i, then such point has $k_i + 1$ separatrices. A direction θ is *completely periodic* if every separatrix in direction θ extends to a saddle connection.
In this case, the set of saddle connections in direction θ separate the surface into a finite number of cylinders, any of which is foliated by periodic orbits of the linear flow ϕ_θ.

Keane directions, Veech dichotomy

A direction θ is a *Keane direction* for X if there is no saddle connection γ in direction θ. Then the flow ϕ_θ is minimal. All but countably many directions are Keane.
On a Veech surface X the flow ϕ_θ is uniquely ergodic whenever θ is Keane, otherwise θ is a completely periodic direction (*Veech dichotomy*, optimal billiard tables).





Both pictures are taken from Curt McMullen's webpage

Diophantine type and translation flows

The *diophantine type* $w(X, \theta) \geqslant 1$ of a Keane direction θ on the surface X is the supremum of those $w \geqslant 1$ such that there exists infinitely many saddle connections γ for the surface X with

$$|\mathrm{Re}(\gamma, \theta)| < \frac{1}{|\mathrm{Im}(\gamma, \theta)|^{w}}.$$

As in the classical case $w(X, \theta) \geqslant 1$ for any X and θ (when $w = 1$ there are infinitely many solutions γ, *Dirichlet's theorem for translation surfaces*).
Jarník Theorem also holds (Marchese, Trevino, Weil, 2015): $\dim_H(\{\theta \mid w(X, \theta) = \eta\}) = \frac{2}{1+\eta}$.

Return times for Keane directions

Let θ be a Keane direction on X, so that the flow ϕ_θ is minimal and the functions $\overline{w}_{\mathrm{hit}}(\phi_\theta, \cdot, \cdot)$, $\underline{w}_{\mathrm{hit}}(\phi_\theta, \cdot, \cdot)$, $\overline{w}_{\mathrm{ret}}(\phi_\theta, \cdot)$ and $\underline{w}_{\mathrm{ret}}(\phi_\theta, \cdot)$ are defined almost everywhere and invariant under ϕ_θ.
An easy geometric argument gives

$$\underline{w}_{\mathrm{ret}}(\phi_\theta, p) \geqslant \frac{1}{w(X, \theta)}.$$

that for any $p \in X$. On the other hand we prove that for Lebesgue almost any point $p \in X$ we have

$$\overline{w}_{\mathrm{ret}}(\phi_\theta, p) = 1.$$

Under the extra assumption that the Lebesgue measure is ergodic for ϕ_θ, for Lebesgue almost any p and p' we have also

$$\underline{w}_{\mathrm{hit}}(\phi_\theta, p, p') = 1.$$

In general, if μ is a non-ergodic invariant probability measure along a Keane direction θ on X, the function $\underline{w}_{\mathrm{hit}}(\phi_\theta, \cdot, \cdot)$ can exhibit a different behavior for the $\mu \times \mu$ generic pair (p, p') (Boshernitzan and Chaika, 2013)

Long hitting times in the stratum $\mathcal{H}(2)$

Theorem (Kim, Marchese, M, 2017) Let X be a surface in $\mathcal{H}(2)$ and let θ be a Keane direction on X. Then for any p, p'

$$\overline{w}_{\mathrm{hit}}(\theta, p, p') \leqslant w(X, \theta)^2.$$

In the general stratum the function $\overline{w}_{\mathrm{hit}}(\phi_\theta, \cdot, \cdot)$ seems to be bounded by a polynomial function of $w(X, \theta)$. In $\mathcal{H}(2)$ there are "many" surfaces and directions for which the bound is sharp.
Theorem (Kim, Marchese, M, 2017) For any $\eta \geqslant 1$ there exists a set of directions $\theta \in \mathbb{E}(X, \eta)$ with

$$\dim_H \left(\mathbb{E}(X, \eta)\right) \geqslant \frac{1}{1+\eta}$$

such that $w(\tan\theta) = \eta$, and for almost any p, p'

$$\overline{w}_{\mathrm{hit}}(\theta, p, p') \geqslant w(\tan\theta)^2 = \eta^2.$$

In particular $\overline{w}_{\mathrm{hit}}(\theta, p, p') = w(\tan\theta)^2 = \eta^2$ for any $X \in \mathcal{H}(2)$, any $\theta \in \mathbb{E}(X, \eta)$ and almost any p, p' in X.

Origamis (aka square-tiled surfaces)

Origamis, which are also know as *square-tiled surfaces*, form a special class of translation surfaces. An origami is a translation surface X tiled by copies of the square $[0,1]^2$. It is a direct consequence of definitions that X is an origami if and only if $\mathrm{Hol}(X) \subset \mathbb{Z}^2$ and the last condition is also equivalent to the existence of a ramified covering $\rho : X \to \mathbb{T}^2$ of the standard torus such that the following conditions are satisfied.

1. The covering is ramified only over the origin $[0] \in \mathbb{T}^2$, where $[0]$ denotes le coset of 0 in $\mathbb{R}^2/\mathbb{Z}^2$.
2. Local inverses of ρ, that is maps $\varphi : U \to X$ defined over simply connected open sets $U \subset \mathbb{T}^2 \setminus \{[0]\}$ such that $\rho \circ \varphi = \mathrm{Id}_U$, are all translations.

A fourth equivalence says that X is an origami if and only if its Veech group $\mathrm{SL}(X)$ shares a common subgroup of finite index with $\mathrm{SL}(2,\mathbb{Z})$. In particular, origamis are all Veech surfaces.

Thanks for your attention!

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And thanks for your patience!