Infinite mixing for one-dimensional maps with an indifferent fixed point

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Quote of the day

254A - Topics in Ergodic theory

"Dynamics on non-compact topological spaces or infinite measure spaces ~~can be quite nasty, and there does not appear to be a useful general theory in these cases~~ is a more complicated topic."

T.Tao, 2008

Theorem (Poincaré Recurrence Theorem)

Let $(X, \mathcal{B}, T, \mu)$ *a measure preserving system and* μ *a probability measure. If* $\mu(A) > 0$ *then there exists* $n \in \mathbb{N}$ *such that* $\mu(A \cap T^{-n}A) > 0$*.*

Consider the map $T : \mathbb{R} \to \mathbb{R}$ such that $T(x) = x + 1$ w.r.t to the Lebesgue measure.

Definition

We say μ is conservative if for all A s.t. $\mu(A) > 0$ we have $A \subseteq \cup_{i \in \mathbb{N}} T^{-i}(A)$.

“Can be quite nasty”

Theorem (Birkhoff)

If $(X, \mathcal{B}, \mu, T)$ is a measure preserving system with $\mu(X) < \infty$ then for any $\varphi \in L^1(\mu)$ it holds

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi\left(T^k x\right) \xrightarrow{n\to\infty} \frac{1}{\mu(X)}\int_X \varphi d\mu$$

for μ-almost every x.

What if $\mu(X) = \infty$?

Theorem (Aaronson)

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system with $\mu(X) = \infty$, and let $\{a_n\}$ be a sequence of positive real numbers. Then, for every $\varphi \in L^1(\mu)$ such that $\varphi \geq 0$ and $\int_X \varphi d\mu > 0$,

$$\limsup_{n\to\infty}\frac{1}{a_n}\sum_{k=0}^{n-1}\varphi\left(T^k x\right) = \infty \ \ a.e \quad or \quad \liminf_{n\to\infty}\frac{1}{a_n}\sum_{k=0}^{n-1}\varphi\left(T^k x\right) = 0 \ \ a.e.$$

Let $S_n(A) = \sum_{i=0}^{n-1} (\chi_A \circ T^i)(A)$. A measure-preserving system is called rationally ergodic if there is a sweep-out set (roughly a set which is hit with probability 1 under the action of T) $A \in \mathcal{B}$ that satisfies a Renyi inequality i.e.

$$\exists M > 0 \text{ s.t. } \forall n \text{ it holds } \int_A S_n(A)^2 d\mu \ \leq \ M \cdot \left(\int_A S_n(A) d\mu \right)^2$$

Theorem (Aaronson)

Every measure-preserving system $(X, \mathcal{B}, \mu, T)$ that is rationally ergodic along a subsequence of iterates admits a subsequence such that for μ-almost every $x \in X$, all $\varphi \in L^1(\mu)$ it holds

$$\frac{1}{N} \sum_{k=1}^{N} \frac{1}{a_{n_k}} \sum_{j=0}^{n_k - 1} \varphi\left(T^j(x)\right) \xrightarrow{n \to \infty} \int_X \varphi d\mu$$

The proof gives the explicit scaling factors $a_{n_k} = \frac{1}{\mu(A)^2} \sum_{j=0}^{n_k - 1} \mu\left(A \cap T^{-j} A\right)$.

What about mixing?

If the preserved measure is finite then one can happily use $\mu(f \circ T^n)\mu(g) \to \mu(f)\mu(g)$

How to extend/replace this definition?

A reasonable option: rational mixing

It corresponds to the existence of a *scaling rate* $(\rho_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \rho_n\, \mu((A \circ T^n)B) = \mu(A)\mu(B)$$
$$\lim_{n\to\infty} \rho_n\, \mu((f \circ T^n)g) = \mu(f)\mu(g),$$

for A, B finite measure sets, equivalently f, g local observables.

Extensive bibliography: Hopf 1937, Krickenberg 1967. Recent times: Aaronson, Thaler, Isola, Melbourne, Theresiu, Zweimuller, Arbieto-Markarian-Pacifico-Soares

Is "patching" with a scaling rate necessary?

Aaronson

"[...] the discussion in [Krengel and Sucheston 1969] indicates that there is no reasonable generalization of mixing."

Framework

- We study maps, either on $(0,1)$ or on $\mathbb{R}^+$ with an indifferent fixed point
- Under mild conditions, they preserve Lebesgue-absolutely continuous infinite measure
- "Mixing" is not due to uniform hyperbolicity
- Extensive Bibliography

Examples

- Farey-like maps
- Pomeau-Manneville maps

- Introduced in Lenci2010
- $T : [0,1] \longrightarrow [0,1]$ has a neutral fixed point in 0 and preserves an infinite measure μ which assigns finite mass to all sets $[a,1]$ for $a \in (0,1)$,

A global observable is any $F \in L^\infty(\mu)$ for which it exists

$$\overline{\mu}(F) := \lim_{a \to 0^+} \frac{1}{\mu([a,1))} \int_a^1 F \, d\mu$$

A local observable is any $g \in L^1(\mu)$.

Key Fact

Pair a global observable and a local observable!

Let $\mathcal{G}$ be the class of global observables, let $\mathcal{L}$ the class of local observables. We briefly recall the definitions of "infinite-volume mixing"

$$\text{(LLM) if } \forall f \in \mathcal{L} \cap \mathcal{G},\ g \in \mathcal{L},\ \lim_{n\to\infty} \mu((f \circ T^n)g) = 0.$$

The dynamical system $((0,1), \mu, T)$ is called *global-local mixing* of type

$$\text{(GLM1) if } \forall F \in \mathcal{G},\ \forall g \in \mathcal{L} \text{ with } \mu(g) = 0,\ \lim_{n\to\infty} \mu((F \circ T^n)g) = 0;$$

$$\text{(GLM2) if } \forall F \in \mathcal{G},\ \forall g \in \mathcal{L},\ \lim_{n\to\infty} \mu((F \circ T^n)g) = \overline{\mu}(F)\mu(g).$$

It is called called *global-global mixing* of type

$$\text{(GGM1) if } \forall F, G \in \mathcal{G},\ \lim_{n\to\infty} \overline{\mu}((F \circ T^n)G) = \overline{\mu}(F)\,\overline{\mu}(G).$$

- $T : (0,1) \longrightarrow (0,1)$, T is a Markov map with its partition $\mathscr{P}$.

(A1) $T|_{I_j}$ possesses an extension $\tau_j : [a_j, a_{j+1}] \longrightarrow [0,1]$ which is bijective and C^2 up to the boundary.

(A2) There exists $\Lambda > 1$ such that $|\tau_j'(x)| \geq \Lambda$, for all $j \geq 1$ and $x \in [a_j, a_{j+1}]$.

(A3) There exists $K > 0$ such that $\dfrac{|\tau_j''(x)|}{|\tau_j'(x)|^2} \leq K$, for all $j \geq 0$ and all x.

(A4) τ_0 is convex with $\tau_0(0) = 0$, $\tau_0'(0) = 1$, and $\tau_0'(x) > 1$, for $x \in (0, a_1]$.

Theorem (Thaler80, Thaler83)

Under the assumptions (A1)-(A4),

- *T preserves an infinite invariant measure μ which is absolutely continuous w.r.t. the Lebesgue measure m and μ is unique up to factors. Moreover, the infinite density $h := d\mu/dm$ is positive and only singular at the indifferent fixed point 0.*
- *T is conservative and exact i.e. A system $(X, \mathcal{B}, \mu, T)$ is exact if and only if for each element S of the tail $\cap_{i=0}^{\infty} T^{-i}\mathcal{B}$ we have $\mu(S) \cdot \mu(X \setminus S) = 0$.*

The given definitions of mixing make sense only if $\overline{\mu}$ is a functional invariant with respect to T, in fact we have

Theorem

Let $T : (0,1) \longrightarrow (0,1)$ *verify (A1)-(A4). For all* $F \in \mathcal{G}$ *and* $n \in \mathbb{N}$, $\overline{\mu}(F \circ T^n)$ *exists and equals* $\overline{\mu}(F)$.

Moreover

Theorem

If $T : (0,1) \to (0,1)$ *verifies (A1)-(A4) is* **(GLM1)** *and* **(LLM)** *but not* **(GGM1)**.

Key Fact

The appropriate strength seems captured by **GLM2**

To prove **GLM2** our strategy relies on constructing a very special local observable which is well behaved with respect to the transfer operator associated to our system. Let $F \in L^\infty$, $g \in L^1$, as usual we define the transfer operator P_T by duality as

$$\int_{(0,1)} (F \circ T) g \, d\mu, = \int_{(0,1)} F(P_T g) d\mu$$

Moreover we define

Definition

A local observable g is *persistently monotonic* if, for all $n \in \mathbb{N}$, $P_T^n g(x)$ is a positive, monotonic function of x.

Theorem (Bonanno, G., Lenci '17)

Let T be a map $(0,1) \longrightarrow (0,1)$ verifying (A1)-(A4). If it admits a persistently monotonic local observable, then T is **(GLM2)**.

For simplicity we consider a Markov partition $\mathscr{P} = \{I_0, I_1\}$ with $I_0 = (0, a_1)$ and $I_1 = (a_1, 1)$. Let's assume that as in Thaler are satisfied. Let ϕ_0, ϕ_1 denote the inverse branches with respect to the map T.

How can we guarantee the existence of a persistently monotonic local observable?

(A5) τ_1 is decreasing.

(A6) $\phi_0 + \phi_1$ is non-decreasing and concave.

(A7) $\phi_0' \, (h \circ \phi_0)/h$ is decreasing and convex.

(A8) $\phi_0'(x) \, h(\phi_0(x)) + \phi_1'(x) \, h(\phi_1(x)) \geq 0$ for all $x \in [0, 1]$, where h is the density of the infinite invariant measure μ given by Thaler theorem.

Theorem (Bonanno, G., Lenci '17)

Let $T : (0, 1) \longrightarrow (0, 1)$ *satisfy assumptions (A1)-(A8) w.r.t. the partition* $\mathscr{P} = \{I_0, I_1\}$*. Then* T *is* **(GLM2)***.*

A family of maps satisfying the assumptions of our Theorem can be defined starting from the inverse branches of the Farey-Parry-Daniels map.

$$F(x) = \begin{cases} \frac{x}{1-x}, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1-x}{x}, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

for which $\phi_0(x) = \frac{x}{1+x}$, $\phi_1(x) = \frac{1}{1+x}$, and $h(x) = \frac{1}{x}$ density of invariant measure. Given $\alpha \in (0,1)$ we consider the map $T_\alpha : [0,1] \longrightarrow [0,1]$ defined using the partition $\mathscr{P} = \{I_0, I_1\}$, with $I_0 = (0, 2^{\alpha-1})$ and $I_1 = (2^{\alpha-1}, 1)$, by

$$\tau_{0,\alpha} = (\phi_0)^{-1} : I_0 \longrightarrow [0,1], \quad \text{with} \quad \phi_{0,\alpha}(x) = \frac{x}{(1+x)^{1-\alpha}}$$

and

$$\tau_{1,\alpha} = (\phi_1)^{-1} : I_1 \longrightarrow [0,1], \quad \text{with} \quad \phi_{1,\alpha}(x) = \frac{1}{(1+x)^{1-\alpha}}.$$

Given a map $T : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with a suitable partition $\mathscr{P}$.
We make the following assumption on T:

(B1) $T|_{I_j}$ is a bijective map onto $\mathbb{R}^+$, and possesses an extension τ_j which, for $j = 0$, is defined on $[a_1, +\infty)$ and, for $j \geq 1$, is defined on $[a_{j+1}, a_j)$ or $(a_{j+1}, a_j]$. τ_j is C^2 up the boundary.

(B2) There exists $\Lambda > 1$ such that $|\tau_j'(x)| \geq \Lambda$, for all $j \geq 1$ and $x \in [a_{j+1}, a_j]$.

(B3) There exists $K > 0$ such that $\dfrac{|\tau_j''(x)|}{|\tau_j'(x)|^2} \leq K$, for all $j \geq 0$ and all x.

(B4) The function $u(x) := x - \tau_0(x)$ is positive, convex and vanishing, as $x \to +\infty$. Furthermore, u'' is decreasing (hence vanishing).

(B5) τ_j increasing for all $j \in \mathcal{J}$

(B6) T preserves the Lebesgue measure m.

(B7) ϕ_j is concave for all $j \in \mathcal{J} \setminus \{0\}$.

Theorem

Under the assumptions (B1)-(B4) and (B6), T is conservative and exact.

The class of global observables is the space $\mathcal{G}$ of all $F \in L^\infty(\mathbb{R}^+, m)$ such that

$$\exists\, \overline{m}(F) := \lim_{a \to +\infty} \frac{1}{a} \int_0^a F\, dm. \tag{1}$$

the class of local observables is $\mathcal{L} := L^1(\mathbb{R}^+, m)$.

Examples

- A bounded periodic F is a global observable, and $\overline{m}(F)$ is the average of F over a period
- Quasi-periodic functions often belong in $\mathcal{G}$, e.g. $F(x) := e^{2\pi i x/\alpha} G(x)$, G is a bounded periodic function (in this case, if the ratio between α and the period of G is irrational, $\overline{m}(F) = 0$; otherwise F is periodic).
- More "random" functions also belong in $\mathcal{G}$: for example, if f is bounded and supported in the interior of $[0, b]$, and $(c_k)_{k\in\mathbb{N}}$ is a bounded sequence which possesses a Cesaro average, then $F(x) := \sum_{k\in\mathbb{N}} c_k f(x - kb)$ is in $\mathcal{G}$.

We follow the same strategy as before by constructing a very special local observable and exploiting its properties with respect to the action of the transfer operator. By doing so we obtain the following:

Theorem (Bonanno, G., Lenci '17)

Let $T : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *verify (B1)-(B7). Then*

- *For all* $F \in \mathcal{G}$ *and* $n \in \mathbb{N}$, $\overline{m}(F \circ T^n)$ *exists and equals* $\overline{m}(F)$.
- T *is* **(GLM1)** *and* **(LLM)***, but not* **(GGM1)**
- T *is* **(GLM2)**

Summary

- Some natural dynamical systems preserve an infinite measure.
- It is still possible to do ergodic theory in this situation (so we do it).
- It is a good idea to pair "global observables" and "local observables".
- Preprint on the Arxiv!

Thank you!