Diffusion along chains of normally hyperbolic cylinders

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Introduction

Arnold Diffusion Problem (1964):
Integrable Hamiltonian systems, of $n > 2$ degrees of freedom, and ‘typical’, subjected to $\varepsilon$-small, ‘generic’ perturbations have trajectories that travel $O(1)$

Informally: Small forcing can produce large effects

Note: Not necessarily a diffusion process
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Example:

- Planar elliptic restricted three-body problem (PERTBP): the primaries move on elliptic orbits of eccentricities $\varepsilon$
- Model: Sun-Jupiter system $\mu = 0.0009537$, $\varepsilon = 0.048$, initial energy $h$ of comet Oterma
- Hamiltonian $H_\varepsilon(x, t) = H_0(x) + \varepsilon H_1(x, t)$, $H_0$ is the Hamiltonian of the PCRTBP
- Theorem [Capiński, M.G., de la Llave, 2014]: There exist $\varepsilon_0 > 0$ and $\rho > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ there exists $x(t)$ s.t. $\|H_0(x(T)) - H_0(x(0))\| > \rho$
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Introduction

- **A priori unstable case:** The unperturbed Hamiltonian possesses a differentiable family of invariant tori that have hyperbolic invariant manifolds
  \[ H_\varepsilon(I, \theta, p, q, t) = \frac{l^2}{2} + \left( \frac{1}{2}p^2 + \cos(q) - 1 \right) + \varepsilon H_1(p, q, l, \theta, t) \]

- **References:** Many

![Diagram of pendulum and rotator](image)
Introduction

- **A priori stable case:** The phase space of the unperturbed Hamiltonian is foliated by Lagrangean invariant tori

\[ H_\varepsilon(l, \theta) = H_0(l) + \varepsilon H_1(l, \theta) \]

- **References:**
  - (Mather, 2002, 2012)
  - (Bernard, Kaloshin and Zhang, 2011), (Kaloshin and Zhang, 2012),
  - (J.-P. Marco, 2016), (M.G., J.-P. Marco, 2016)
Objectives

▶ Provide a geometric mechanism of diffusion in the a priori stable case
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▶ Steps:
  
  I. Resonances determine chains of normally hyperbolic cylinders (NHIC's) (J.-P. Marco, 2016)
  
  II. There exist diffusing orbits along the chains, under certain conditions (M.G. and J.-P. Marco, 2016)
  
  III. Those conditions satisfy Mather’s cusp residual condition (J.-P. Marco, 2016)
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- I. Resonances determine chains of normally hyperbolic cylinders (NHIC’s) (J.-P. Marco, 2016)
- II. There exist diffusing orbits along the chains, under certain conditions (M.G. and J.-P. Marco, 2016)
- III. Those conditions satisfy *Mather’s cusp residual condition* (J.-P. Marco, 2016)
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- Approach for II (joint work with J.-P. Marco):
  - Geometric – goes back to Birkhoff’s theory on connecting orbits; related approaches (Moeckel, 2002), (Le Calvez, 2007), (M.G. and C. Robinson, 2013)
  - Low dimensional – diffusing orbits are obtained for 2-dimensional dynamics
  - Constructive – diffusing orbits can be found explicitly (via an algorithm)
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Set-up and main results

Theorem

- **Given:**
  - \( H_\varepsilon(I, \theta) = H_0(I) + \varepsilon H_1(I, \theta), \)
  - \( C^2 \)-Hamiltoninan, \((I, \theta) \in A^3 = \mathbb{R}^3 \times T^3\)
  - \( H_0 \) strictly convex and super-linear
Set-up and main results

Theorem

▶ Given:
  ▶ \( H_\varepsilon(l, \theta) = H_0(l) + \varepsilon H_1(l, \theta) \),
  \( C^2 \)-Hamiltoninan, \((l, \theta) \in A^3 = \mathbb{R}^3 \times T^3 \)
  ▶ \( H_0 \) strictly convex and super-linear

▶ Then:
  ▶ for every \( O_1, O_2, \ldots, O_n \) open sets in \( \mathbb{R}^3 \),
    \( h \) regular value of \( H_0 \), s.t.
    \( O_j \cap \{H = h\} \neq \emptyset \),
    there exists a set \( \mathcal{R} \) such that for each
    \( \varepsilon H_1 \in \mathcal{R} \) there exists \( \Phi_\varepsilon^x(t) \) in \( \{H_\varepsilon = h\} \)
    with \( \Phi_\varepsilon^{x_j}(x) \in O_j \times T^3 \) for some \( t_j \),
    \( j = 1, \ldots, n \)
Set-up and main results

- Description of the set $\mathcal{R}$ — Mather’s cusp residual condition:

  There exists an open-dense set $O \subset S_1 := \{ H_1 : \|H_1\| = 1 \}$

  There exists a lower semi-continuous function $\varepsilon_0 : S_1 \to \mathbb{R}$ with $\varepsilon_0 > 0$ on $O$

  The set $\mathcal{R}$ claimed in the theorem is open and dense in $\{ \varepsilon H_1 : 0 < \varepsilon < \varepsilon_0 (H_1) \}$, $H_1 \in O$

  Note:

  Certain directions $H_1 \in S_1$ are 'bad', i.e., lead to $H_\varepsilon$ integrable

  Example:

  $H_0 = \frac{1}{2}(I_{21} + I_{22})$

  $H_1 = \cos(\theta_3) \Rightarrow H_\varepsilon = \frac{1}{2}(I_{21} + I_{22}) + \left( I_{23} + \varepsilon \cos(\theta_3) \right)$
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- Description of the set $\mathcal{R}$ — Mather’s cusp residual condition:
  - There exists an open-dense set $\mathcal{O} \subset S^1 := \{ H_1 : \| H_1 \| = 1 \}$
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- Example:
  $H_0 = \frac{1}{2} \left( I_{1,1} + I_{2,2} + I_{3,3} \right)$
  $H_1 = \cos(\theta_3) \Rightarrow H_\varepsilon = \frac{1}{2} \left( I_{1,1} + I_{2,2} \right) + \left( I_{3,3} + \varepsilon \cos(\theta_3) \right)$
Set-up and main results

- Description of the set \( \mathcal{R} \) — Mather’s cusp residual condition:
  - There exists an open-dense set \( \mathcal{O} \subset S^1 := \{H_1 : \|H_1\| = 1\} \)
  - There exists a lower semi-continuous function \( \varepsilon_0 : S^1 \to \mathbb{R} \) with \( \varepsilon_0 > 0 \) on \( \mathcal{O} \)
  - The set \( \mathcal{R} \) claimed in the theorem is open and dense in \( \{\varepsilon H_1 : 0 < \varepsilon < \varepsilon_0(H_1), H_1 \in \mathcal{O}\} \)

Note:
- Certain directions \( H_1 \in S^1 \) are ‘bad’, i.e., lead to integrable
- Example:
  \[
  H_0 = \frac{1}{2}(I_2^1 + I_2^2 + I_2^3) \]
  \[
  H_1 = \cos(\theta_3) \Rightarrow \varepsilon H_1 = \frac{1}{2}(I_2^1 + I_2^2) + (I_2^3 + \varepsilon \cos(\theta_3))
  \]
Set-up and main results

- Description of the set $\mathcal{R}$ — Mather’s cusp residual condition:
  - There exists an open-dense set $\mathcal{O} \subset S^1 := \{H_1 : \|H_1\| = 1\}$
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- Note:
  - Certain directions $H_1 \in S^1$ are ‘bad’, i.e., lead to $H_\varepsilon$ integrable
  - Example:
    
    $H_0 = \frac{1}{2}(l_1^2 + l_2^2 + l_3^2)$
    $H_1 = \cos(\theta_3) \rightsquigarrow$
    $H_\varepsilon = \frac{1}{2}(l_1^2 + l_2^2) + \left(\frac{l_3^2}{2} + \varepsilon \cos(\theta_3)\right)$
Case $\varepsilon = 0$:

- In action space $\mathbb{R}^3$ – resonance relations
  \[ k_1\omega_1(l) + k_2\omega_2(l) + k_3\omega_3(l) = 0, \]
  with $(k_1, k_2, k_3) \in \mathbb{Z}^3 \setminus \{0\}$, where
  \[ \omega(l) = \partial H_0 / \partial l \]
- Simple resonance $\leadsto$ surface in $\mathbb{R}^3$;
- Double resonance $\leadsto$ intersection of two resonant surfaces $\leadsto$ curve in $\mathbb{R}^3$
- \( \{H_0 = h\} \cap \mathbb{R}^3 \leadsto \) 2-sphere – simple resonances $\leadsto$ curves;
  double resonances $\leadsto$ points
Geometric structures

Case $\varepsilon = 0$:

- In action space $\mathbb{R}^3$ – resonance relations
  
  \[ k_1 \omega_1(I) + k_2 \omega_2(I) + k_3 \omega_3(I) = 0, \]

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- Simple resonance $\leadsto$ surface in $\mathbb{R}^3$;

- Double resonance $\leadsto$ intersection of two resonant surfaces $\leadsto$ curve in $\mathbb{R}^3$

\[ \{H_0 = h\} \cap \mathbb{R}^3 \leadsto 2\text{-sphere} \quad \text{simple resonances} \leadsto \text{curves}; \quad \text{double resonances} \leadsto \text{points} \]
Case $\varepsilon > 0$ small

- Distinguish between ‘strong double resonances’ and ‘weak double (almost simple) resonances’
- Along (almost) simple resonances the perturbation determines NHIC’s $\approx \mathbb{T}^2 \times [a, b]$
- Assume $H_0 = \frac{1}{2}(l_1^2 + l_2^2 + l_3^2)$
- Set $H_0 = h$ and $l_3 = 0$:
  \[
  \dot{\theta}_1 = l_1 + \varepsilon \frac{dH_1}{dl_1} \\
  \dot{\theta}_2 = l_2 + \varepsilon \frac{dH_1}{dl_2} \\
  \dot{\theta}_3 \approx \varepsilon \frac{dH_1}{dl_3}
  \]
- $\Rightarrow \theta_1, \theta_2$-fast angles, $\theta_3$-slow angle
- $V(\theta_3) = \int_{\mathbb{T}^2} H_1(\theta_1, \theta_2, \theta_3) d\theta_1 d\theta_2$
- $H_\varepsilon \approx \frac{1}{2}(l_1^2 + l_2^2) + \left(\frac{1}{2}l_3^2 + \varepsilon V(\theta_3)\right)$
Geometric structures

- In \( \{ H_\varepsilon = h \} \) – there exist chains of 3-dimensional cylinders along the simple resonances that get close to one another near double resonances.

- At ‘strong’ double resonances – singular cylinders, that can be ‘continued’ to non-singular ones.
Geometric structures

- Chains of cylinders:
  - \( C_1, C_2, \ldots, C_k \) in \( \{ H_\varepsilon = h \} \)
  - \( C_i \cong \mathbb{T}^2 \times [a_i, b_i] \), \( C^1 \)-smooth
  - \( C_i \) normally hyperbolic invariant manifold with boundary
  - \( W^u(C_i) \cap W^s(C_i) \neq \emptyset \) and \( W^u(C_i) \cap W^s(C_{i+1}) \neq \emptyset \)
  - For each \( O_j \) there exist a \( C_i \) and an ‘essential torus’ \( \mathcal{T} \) in \( C_i \) s.t. \( O_j \cap \mathcal{T} \neq \emptyset \)

- Want to prove:
  - Given \( \delta > 0 \), there exists \( \Phi^\varepsilon_t(x) \), s.t. for each \( i \) and each essential torus \( \mathcal{T} \) in \( C_i \), there is \( t_x \) s.t. \( \Phi^\varepsilon_{t_x}(x) \) in a \( \delta \)-neighborhood of \( \mathcal{T} \)
Geometric structures

- Scattering maps associated to $W^u(C_i) \cap W^s(C_i) \neq \emptyset$ and $W^u(C_i) \cap W^s(C_{i+1}) \neq \emptyset$

  - $\psi_i(x_-) = x_+$ iff $W^u(x_-) \cap W^s(C_i)$ and $W^s(x_+) \cap W^u(C_i)$ at $x \rightsquigarrow$
  
  - $\psi_i : \text{dom}(\psi_i) \subseteq C_i \rightarrow \text{im}(\psi_i) \subseteq C_i$

  - $\psi_i(x_-) = x_+$ iff $W^u(x_-) \cap W^s(C_{i+1})$ and $W^s(x_+) \cap W^u(C_i)$ at $x \rightsquigarrow$
  
  - $\psi_{i+1}^i : \text{dom}(\psi_{i+1}^i) \subseteq C_i \rightarrow \text{im}(\psi_{i+1}^i) \subseteq C_{i+1}$

- Collection of all possible scattering maps — scattering relation (which can be iterated)
For each $C_i$ there exists $A_i \simeq \mathbb{T}^1 \times [a_i, b_i]$ global surface of section$^1$ and $\phi_i : A_i \to A_i$ first return map s.t.:

- $\partial_\bullet A_i$, $\partial^\bullet A_i$ are $\phi_i$-invariant and $\phi_i$-minimal
- $\phi_i$ is a ‘special twist map’ on $A_i$

$\psi_i, \psi_{i,i+1}$ induce scattering maps on $A_i$’s, which are symplectic (Delshams, de la Llave, Seara, 2008)

$\psi_{i,i+1}$ maps a neighborhood of $\partial^\bullet A_i$ onto a neighborhood of $\partial_\bullet A_{i+1}$

- on each cylinder there are two dynamics – inner dynamics $\phi_i$ and outer dynamics $\psi_i$
- in addition, there is outer dynamics from one cylinder to the next

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$^1$follows from the convexity of the unperturbed system
Differences from the a priori unstable case

- The inner dynamics on $C_i$ is not close to integrable
- The scattering maps $\psi_i, \psi_i^{i+1}$ cannot be computed via perturbation theory
Main result

Theorem (M.G. and J.-P. Marco, 2016)

For a given annulus $A_i$, if $\phi_i$ and $\psi_i$ are in ‘general position’ relative to $A_i$ then there exists a pseudo-orbit of $\{\phi_i, \psi_i\}$ from a neighborhood of $\partial \bullet A_i$ to a neighborhood of $\partial \bullet A_i$, which visits a $\delta$-neighborhood of every $\Gamma \in \text{Ess}(\phi_i)$.

- Notation: $\text{Ess}(\phi_i) =$ essential (i.e., homotopically non-trivial) invariant circles of $\phi_i$ on $A_i$
- Similar idea to (Moeckel, 2002) with one significant difference: $\psi_i$ not defined on the whole annulus

\(^2\)in particular, if no circle in $\text{Ess}(\phi_i)$ is left invariant under $\psi_i$
Main result

Corollary

For the whole system of annuli \( \{A_i\}_{i=1,...,k} \) there exists a pseudo-orbit of \( \{\phi_i, \psi_i, \psi^{i+1}_i\} \) from a neighborhood of \( \partial\bullet A_1 \) to a neighborhood of \( \partial\bullet A_k \), which visits a \( \delta \)-neighborhood of every \( \Gamma \in \text{Ess}(\phi_i) \) for every \( i \)

Corollary

Pseudo-orbits of \( \{\phi_i, \psi_i, \psi^{i+1}_i\} \) can be shadowed by true orbits of \( \Phi^\varepsilon_t \) – follows from a Shadowing Lemma in (M.G., de la Llave, Seara, 2014)

- The Shadowing Lemma mentioned above uses Poincaré Recurrence Theorem
- In conclusion, there exist trajectories of \( \Phi^\varepsilon_t \) that ‘diffuse’ along the chain of cylinders \( C_1, \ldots, C_k \) and visit a \( \delta \)-neighborhood of each essential torus \( T \)
- No control on diffusion time
Sketch of the proof of the main result

What do we mean by a ‘special twist map’?

- each $\Gamma \in \text{Ess}(\phi_i)$ has irrational rotation number
- each $\Gamma \in \text{Ess}(\phi_i)$ is either the upper bound of a Birkhoff Zone of Instability (BZI) or is a limit from below, in the Hausdorff topology, of essential circles $\Gamma_n \prec \Gamma$ in $\text{Ess}(\phi_i)$
- these conditions are generic (Herman,1983)
Sketch of the proof of the main result

- What do we mean by the inner and the outer dynamics being in ‘general position’?
  - Assumptions on the outer dynamics
    - **Splitting property:** for each $\Gamma \in \text{Ess}(\phi_i)$ which is a limit from below of $\Gamma_n$’s, there exists a splitting arc $A \subset \Gamma^-$ ($= \text{the region in } A_i \text{ below } \Gamma$) with $\psi_i(A) \subseteq \Gamma$
    - **Right (left) triangle property:** for each $\Gamma$ which is the upper bound of a BZI there exists a right (left) triangle $T \subseteq \Gamma^-$ with $\psi_i(T) \subseteq \Gamma^+$ ($= \text{the region in } A_i \text{ above } \Gamma$)
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Sketch of the proof of the main result

- Constructive version:
  - apply the Birkhoff procedure to $\phi$ until you get an essential circle $\Gamma$
  - apply the scattering map $\psi$ to $\Gamma$ and apply again the Birkhoff procedure to $\phi$
  - key idea – essential circles obtained via this procedure cannot accumulate – due to the right (left) triangle property

- Lemma: If $T$ is a right (left) triangle for $\Gamma$ and $\Gamma' \prec \Gamma$ is sufficiently close to $\Gamma$, then there exists a right (left) triangle $T'$ for $\Gamma'$ such that $\psi(T') \cap \Gamma^+ \neq \emptyset$
Sketch of the proof of the main result

How to use ‘left triangles’ instead of ‘right triangles’

- In the above constructions take negative iterates $\phi^{-n}$, $n > 0$ of the twist map $\phi$, rather than positive ones.

- Key ingredient: positive iterates of a vertical line under $\phi$ — right tilted curves; negative iterates of a vertical line under $\phi^{-1}$ — left tilted curves.

- The orbit of $N_\bullet$ under the polysystem $\{\phi, \psi\}$ is dense in the orbit of $N_\bullet$ under the polysystem $\{\phi, \phi^{-1}, \psi\}$ – use Poincaré Recurrence Theorem.
Sketch of the proof of the main result

Shadowing Lemma for NHIM’s (M.G., de la Llave, Seara, 2014)

Assume that $f : M \to M$ is a smooth map, $\mathcal{C} \subseteq M$ is a NHIM, $\psi$ is a scattering map on some open subset of $\mathcal{C}$, and $\phi = f|_{\mathcal{C}}$.

Then, for every $\delta > 0$ there exist $n^* \in \mathbb{N}$, and $m_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, s.t., for every pseudo-orbit $\{y_i\}_{i \geq 0}$ in $\mathcal{C}$ of the form

$$y_{i+1} = \phi^{m_i} \circ \psi^\Gamma \circ \phi^{n_i}(y_i),$$

with $n_i \geq n^*$ and $m_i \geq m_i^*(n_0, \ldots, n_{i-1}, n_i, m_0, \ldots, m_{i-1})$, there exists an orbit $\{z_i\}_{i \geq 0}$ of $f$ in $M$ such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i), \text{ and } d(z_i, y_i) < \delta.$$

**Remark:** No assumption on the ‘inner dynamics’ given by $\phi$