Diffusion along chains of normally hyperbolic cylinders

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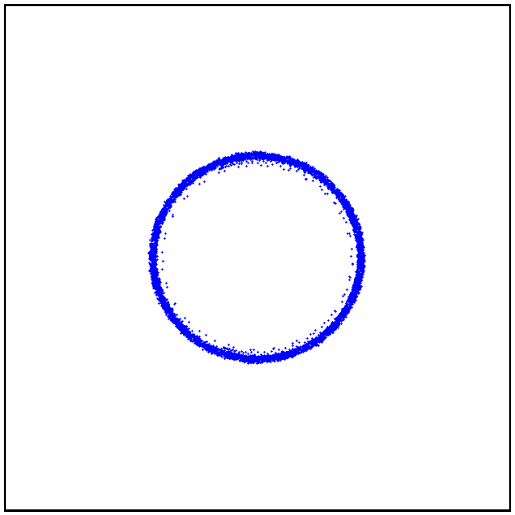
Outline

Introduction

- Arnold Diffusion Problem (1964): Integrable Hamiltonian systems, of $n > 2$ degrees of freedom, and 'typical', subjected to ε-small, 'generic' perturbations have trajectories that travel $O(1)$
- Informally: Small forcing can produce large effects
- Note: Not necessarily a diffusion process



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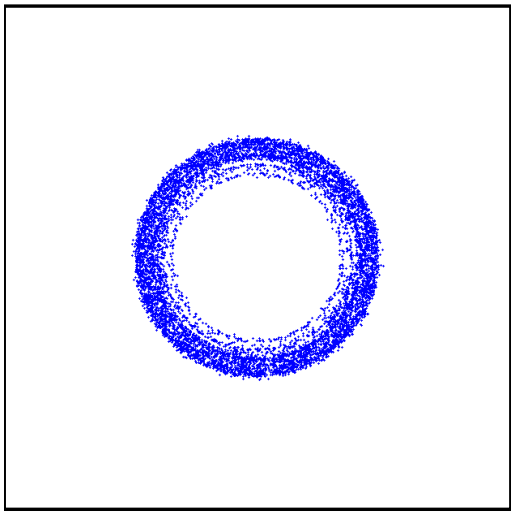

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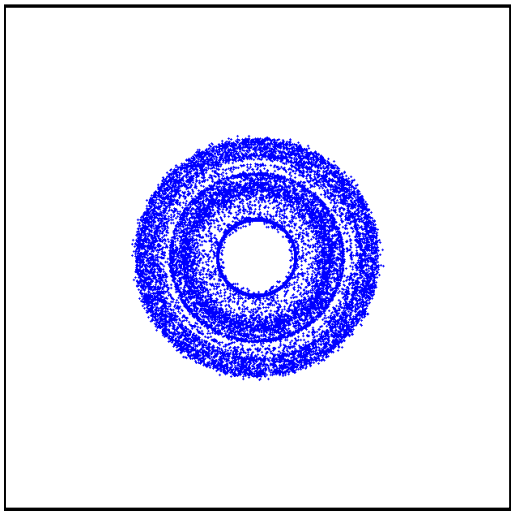

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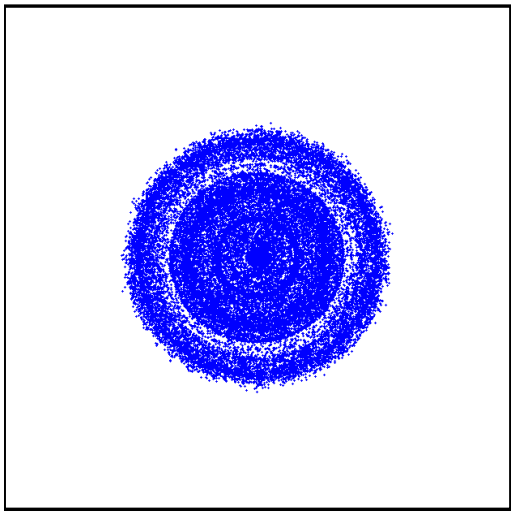

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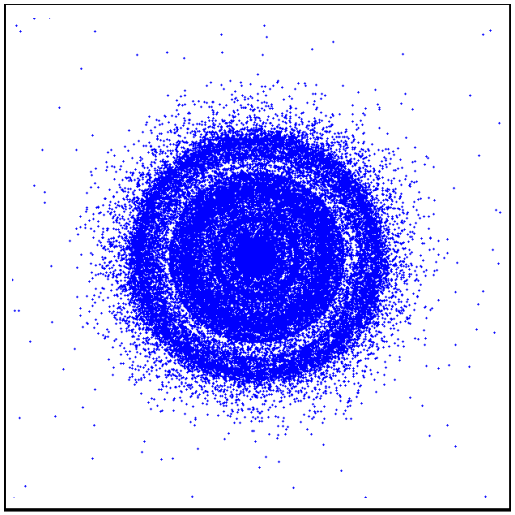

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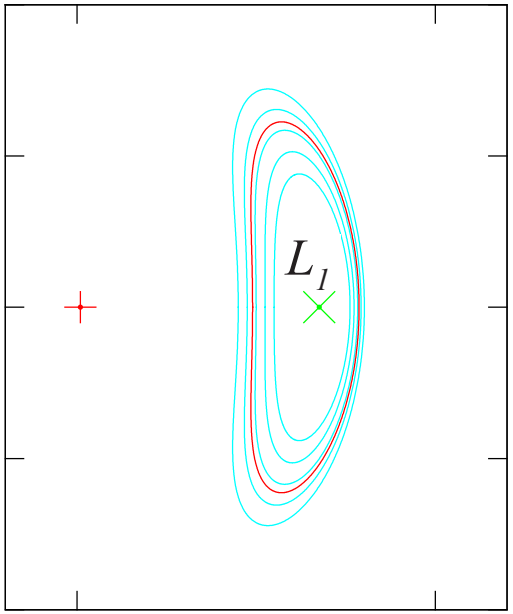
- Example:
  - Planar elliptic restricted three-body problem (PERTBP): the primaries move on elliptic orbits of eccentricities ε
  - Model: Sun-Jupiter system $\mu = 0.0009537$, $\varepsilon = 0.048$, initial energy h of comet Oterma
  - Hamiltonian $H_\varepsilon(\mathbf{x}, t) = H_0(\mathbf{x}) + \varepsilon H_1(\mathbf{x}, t)$, H_0 is the Hamiltonian of the PCRTBP
  - Theorem [Capiński, M.G., de la Llave, 2014]: There exist $\varepsilon_0 > 0$ and $\rho > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ there exists $\mathbf{x}(t)$ s.t. $\|H_0(\mathbf{x}(T)) - H_0(\mathbf{x}(0))\| > \rho$



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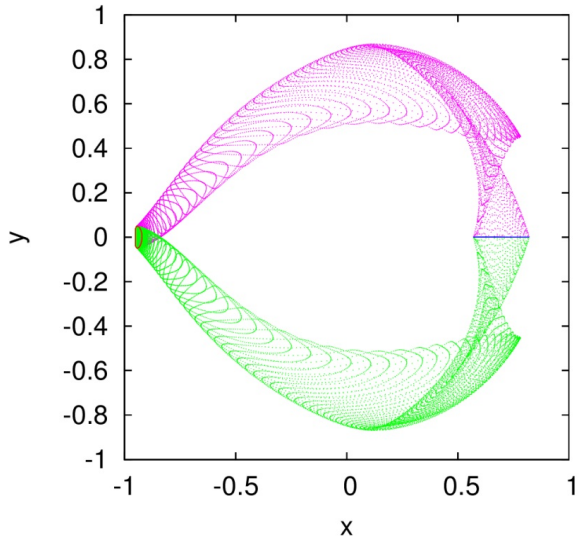

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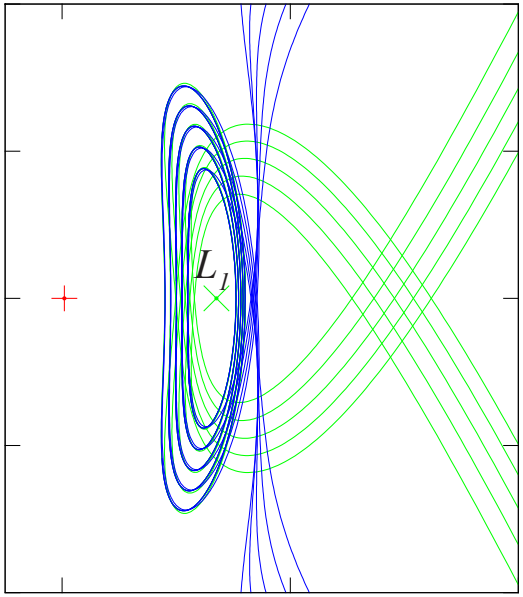
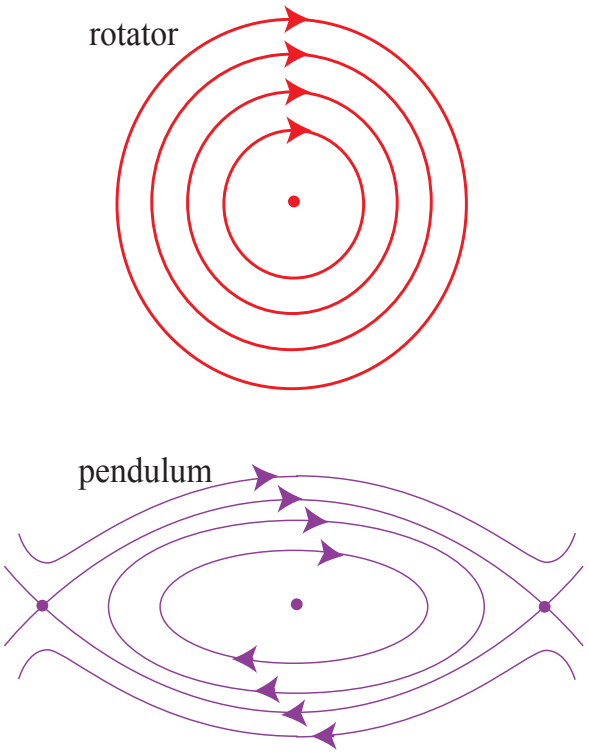


Introduction

- A priori unstable case: The unperturbed Hamiltonian possesses a differentiable family of invariant tori that have hyperbolic invariant manifolds $H_\varepsilon(I,\theta,p,q,t) = \frac{I^2}{2} + \left(\frac{1}{2}p^2 + \cos(q) - 1\right) + \varepsilon H_1(p,q,I,\theta,t)$
- References: Many

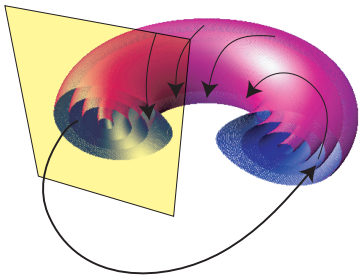


Introduction

- A priori stable case: The phase space of the unperturbed Hamiltonian is foliated by Lagrangean invariant tori

$$H_\varepsilon(I,\theta) = H_0(I) + \varepsilon H_1(I,\theta)$$

- References:
  - (Mather,2002,2012)
  - (Bernard, Kaloshin and Zhang,2011), (Kaloshin and Zhang,2012),
  - (C.-Q. Cheng,2013), (C.-Q. Cheng and J. Xue,2015),
  - (J.-P. Marco,2016), (M.G.,J.-P. Marco,2016)



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  - II. There exist diffusing orbits along the chains, under certain conditions (M.G. and J.-P. Marco,2016)
  - III. Those conditions satisfy *Mather's cusp residual condition* (J.-P. Marco,2016)

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  - Low dimensional – diffusing orbits are obtained for 2-dimensional dynamics
  - Constructive – diffusing orbits can be found explicitly (via an algorithm)

Set-up and main results

Theorem

- Given:
  - $H_\varepsilon(I,\theta) = H_0(I) + \varepsilon H_1(I,\theta)$,
    C^2-Hamiltoninan, $(I,\theta) \in \mathbb{A}^3 = \mathbb{R}^3 \times \mathbb{T}^3$
  - H_0 strictly convex and super-linear

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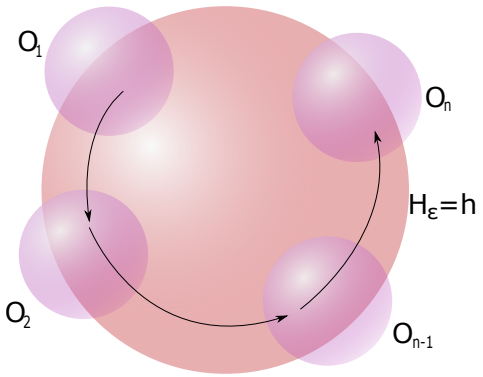
  - H_0 strictly convex and super-linear
- Then:
  - for every $O_1, O_2, \dots O_n$ open sets in $\mathbb{R}^3$,
    h regular value of H_0, s.t.
    $O_j \cap \{H = h\} \neq \emptyset$,
    there exists a set $\mathcal{R}$ such that for each
    $\varepsilon H_1 \in \mathcal{R}$ there exists $\Phi_t^\varepsilon(\mathbf{x})$ in $\{H_\varepsilon = h\}$
    with $\Phi_{t_j}^\varepsilon(\mathbf{x}) \in O_j \times \mathbb{T}^3$ for some t_j,
    $j = 1, \dots, n$



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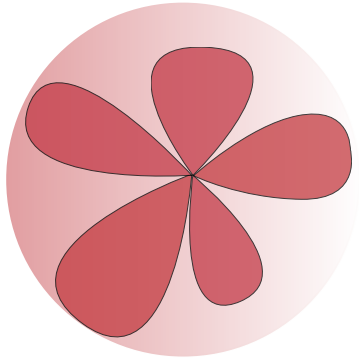

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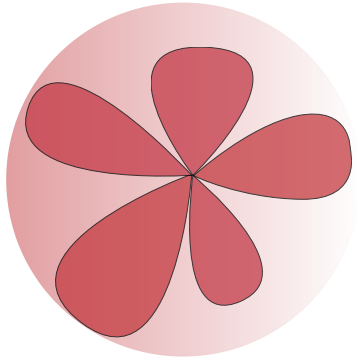

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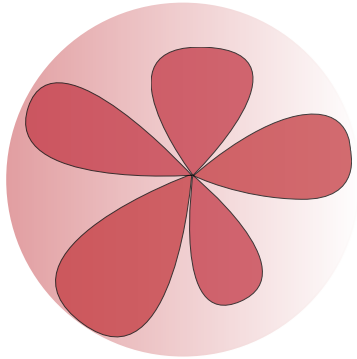

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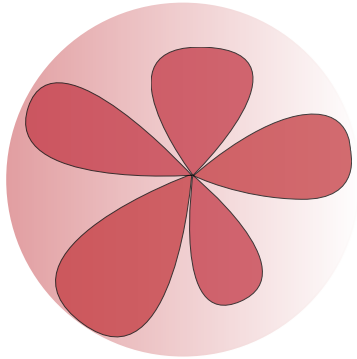
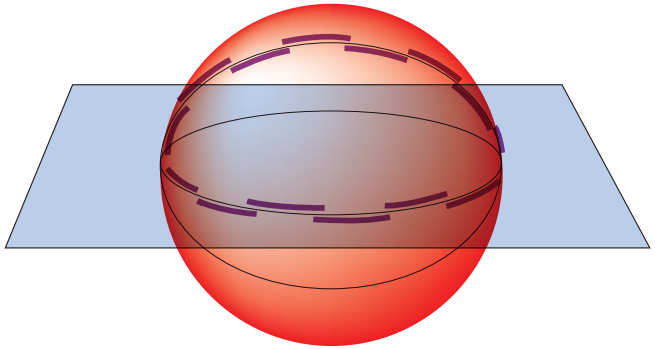
- Note:
  - Certain directions $H_1 \in \mathcal{S}^1$ are 'bad', i.e., lead to H_ε integrable
  - Example:
    $H_0 = \frac{1}{2}(I_1^2 + I_2^2 + I_3^2)$
    $H_1 = \cos(\theta_3) \rightsquigarrow$
    $H_\varepsilon = \frac{1}{2}(I_1^2 + I_2^2) + \left(\frac{I_3^2}{2} + \varepsilon \cos(\theta_3)\right)$



Geometric structures

- Case $\varepsilon = 0$:
  - In action space $\mathbb{R}^3$ – resonance relations $k_1\omega_1(I) + k_2\omega_2(I) + k_3\omega_3(I) = 0$, with $(k_1, k_2, k_3) \in \mathbb{Z}^3 \setminus \{0\}$, where $\omega(I) = \partial H_0/\partial I$
  - Simple resonance $\rightsquigarrow$ surface in $\mathbb{R}^3$;
  - Double resonance = intersection of two resonant surfaces $\rightsquigarrow$ curve in $\mathbb{R}^3$
  - $\{H_0 = h\} \cap \mathbb{R}^3 \rightsquigarrow$ 2-sphere – simple resonances $\rightsquigarrow$ curves; double resonances $\rightsquigarrow$ points



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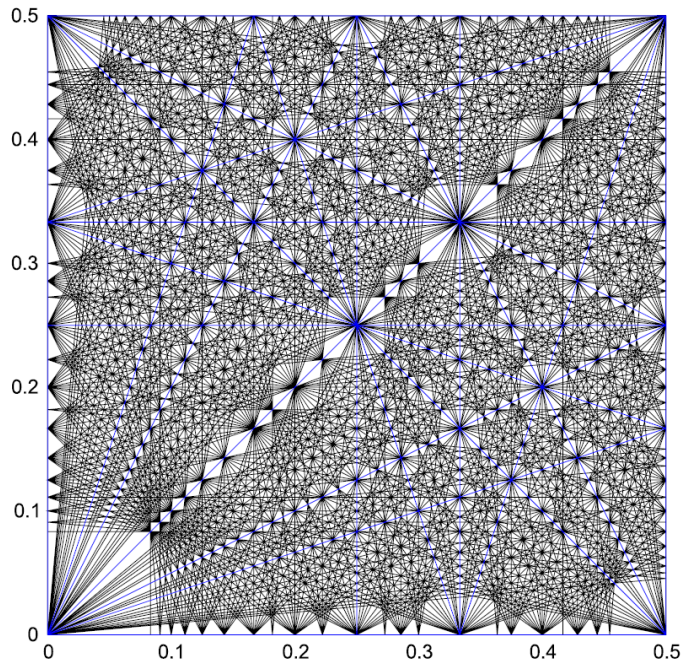
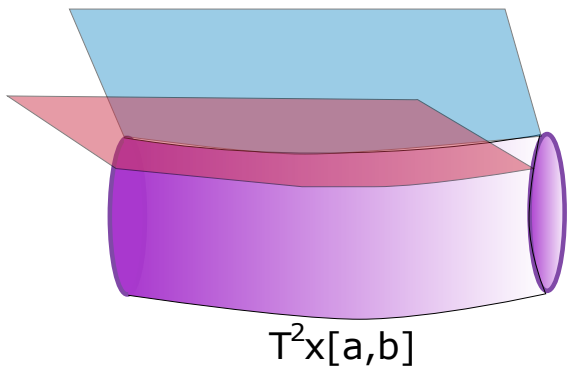


Credit: Gelfreich, Simo, Vieiro

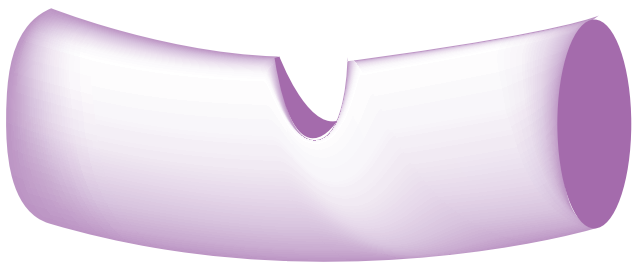
Geometric structures

- Case $\varepsilon > 0$ small
  - Distinguish between 'strong double resonances' and 'weak double (almost simple) resonances'
  - Along (almost) simple resonances the perturbation determines NHIC's $\approx \mathbb{T}^2 \times [a, b]$
    - Assume $H_0 = \frac{1}{2}(I_1^2 + I_2^2 + I_3^2)$
    - Set $H_0 = h$ and $I_3 = 0$:
      $\dot{\theta}_1 = I_1 + \varepsilon \frac{dH_1}{dI_1}$
      $\dot{\theta}_2 = I_2 + \varepsilon \frac{dH_1}{dI_2}$
      $\dot{\theta}_3 \approx \varepsilon \frac{dH_1}{dI_3}$
      $\rightsquigarrow$ θ_1, θ_2-fast angles, θ_3-slow angle
    - $V(\theta_3) = \int_{\mathbb{T}^2} H_1(\theta_1, \theta_2, \theta_3) d\theta_1 d\theta_2$
    - $H_\varepsilon \approx \frac{1}{2}(I_1^2 + I_2^2) + \left(\frac{1}{2} I_3^2 + \varepsilon V(\theta_3)\right)$



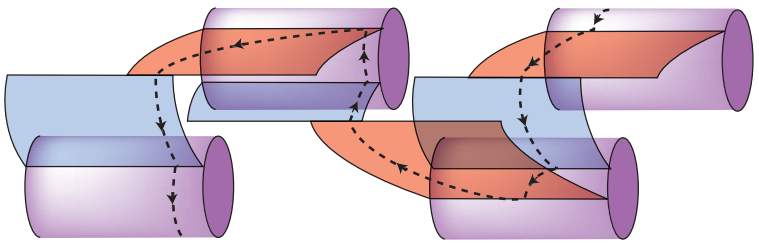
Geometric structures

- In $\{H_\varepsilon = h\}$ – there exist chains of 3-dimensional cylinders along the simple resonances that get close to one another near double resonances
- At 'strong' double resonances – singular cylinders, that can be 'continued' to non-singular ones



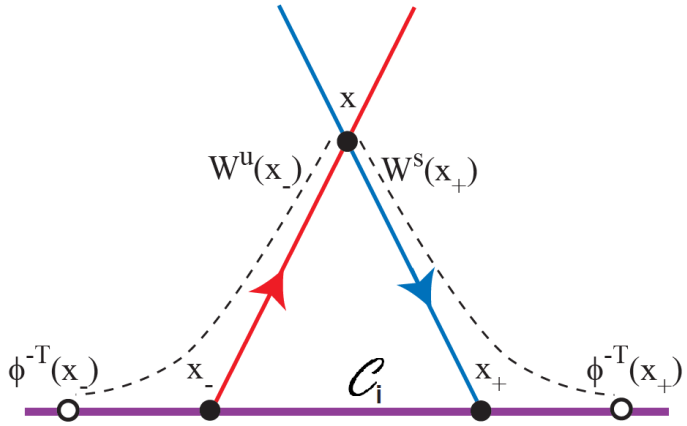
Geometric structures

- Chains of cylinders:
  - $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ in $\{H_\varepsilon = h\}$
  - $\mathcal{C}_i \simeq \mathbb{T}^2 \times [a_i, b_i]$, C^1-smooth
  - $\mathcal{C}_i$ normally hyperbolic invariant manifold with boundary
  - $W^u(\mathcal{C}_i) \cap W^s(\mathcal{C}_i) \neq \emptyset$ and $W^u(\mathcal{C}_i) \cap W^s(\mathcal{C}_{i+1}) \neq \emptyset$
  - For each O_j there exist a $\mathcal{C}_i$ and an 'essential torus' $\mathcal{T}$ in $\mathcal{C}_i$ s.t. $O_j \cap \mathcal{T} \neq \emptyset$
- Want to prove:
  - Given $\delta > 0$, there exists $\Phi_t^\varepsilon(\mathbf{x})$, s.t. for each i and each essential torus $\mathcal{T}$ in $\mathcal{C}_i$, there is t_x s.t. $\Phi_{t_x}^\varepsilon(\mathbf{x})$ in a δ-neighborhood of $\mathcal{T}$



Geometric structures

- Scattering maps associated to $W^u(\mathcal{C}_i) \cap W^s(\mathcal{C}_i) \neq \emptyset$ and $W^u(\mathcal{C}_i) \cap W^s(\mathcal{C}_{i+1}) \neq \emptyset$
  - $\psi_i(x_-) = x_+$ iff $W^u(x_-) \pitchfork W^s(\mathcal{C}_i)$ and $W^s(x_+) \pitchfork W^u(\mathcal{C}_i)$ at $x \rightsquigarrow$ $\psi_i : \mathrm{dom}(\psi_i) \subseteq \mathcal{C}_i \to \mathrm{im}(\psi_i) \subseteq \mathcal{C}_i$
  - $\psi_i(x_-) = x_+$ iff $W^u(x_-) \pitchfork W^s(\mathcal{C}_{i+1})$ and $W^s(x_+) \pitchfork W^u(\mathcal{C}_i)$ at $x \rightsquigarrow$ $\psi_i^{i+1} : \mathrm{dom}(\psi_i^{i+1}) \subseteq \mathcal{C}_i \to \mathrm{im}(\psi_i^{i+1}) \subseteq \mathcal{C}_{i+1}$
- Collection of all possible scattering maps — scattering relation (which can be iterated)



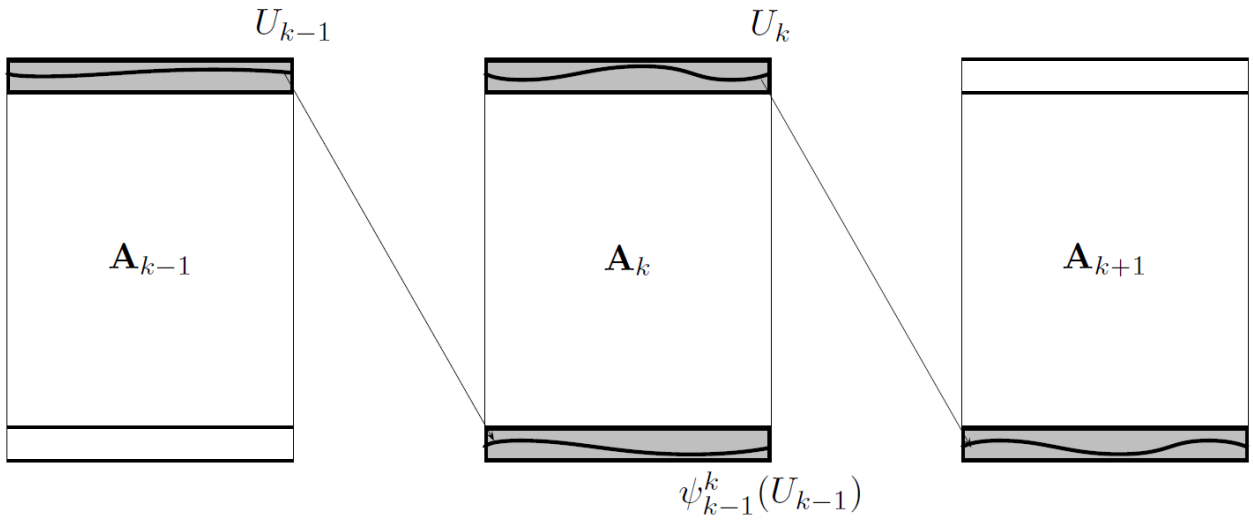
Geometric structures

- For each $\mathcal{C}_i$ there exists $A_i \simeq \mathbb{T}^1 \times [a_i, b_i]$ global surface of section[1] and $\phi_i : A_i \to A_i$ first return map s.t.:
  - $\partial_\bullet A_i$, $\partial^\bullet A_i$ are ϕ_i-invariant and ϕ_i-minimal
  - ϕ_i is a 'special twist map' on A_i
- ψ_i, $\psi_{i,i+1}$ induce scattering maps on A_i's, which are symplectic (Delshams, de la Llave, Seara, 2008)
- $\psi_{i,i+1}$ maps a neighborhood of $\partial^\bullet A_i$ onto a neighborhood of $\partial_\bullet A_{i+1}$
- on each cylinder there are two dynamics – inner dynamics ϕ_i and outer dynamics ψ_i
- in addition, there is outer dynamics from one cylinder to the next

[1] follows from the convexity of the unperturbed system

Differences from the a priori unstable case

- The inner dynamics on $\mathcal{C}_i$ is not close to integrable
- The scattering maps ψ_i, ψ_i^{i+1} cannot be computed via perturbation theory



Main result

Theorem (M.G. and J.-P. Marco, 2016)

For a given annulus A_i, if ϕ_i and ψ_i are in 'general position'[2] relative to A_i then there exists a pseudo-orbit of $\{\phi_i, \psi_i\}$ from a neighborhood of $\partial_\bullet A_i$ to a neighborhood of $\partial^\bullet A_i$, which visits a δ-neighborhood of every $\Gamma \in \mathrm{Ess}(\phi_i)$.

- Notation: $\mathrm{Ess}(\phi_i)$ = essential (i.e., homotopically non-trivial) invariant circles of ϕ_i on A_i
- Similar idea to (Moeckel,2002) with one significant difference: ψ_i not defined on the whole annulus

[2] in particular, if no circle in $\mathrm{Ess}(\phi_i)$ is left invariant under ψ_i

Main result

Corollary

For the whole system of annuli $\{A_i\}_{i=1,\dots,k}$ there exists a pseudo-orbit of $\{\phi_i, \psi_i, \psi_i^{i+1}\}$ from a neighborhood of $\partial_\bullet A_1$ to a neighborhood of $\partial^\bullet A_k$, which visits a δ-neighborhood of every $\Gamma \in \mathrm{Ess}(\phi_i)$ for every i

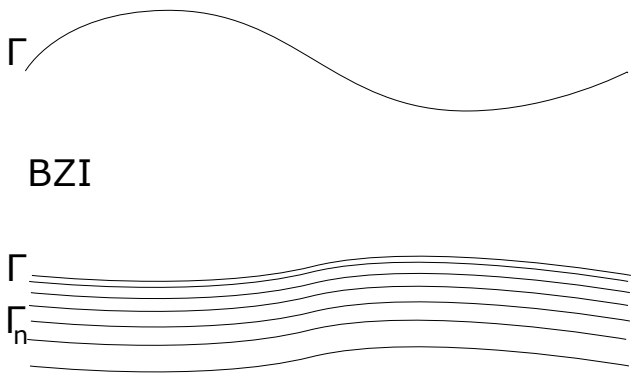
Corollary

Pseudo-orbits of $\{\phi_i, \psi_i, \psi_i^{i+1}\}$ can be shadowed by true orbits of Φ_t^ε – follows from a Shadowing Lemma in (M.G., de la Llave, Seara,2014)

- The Shadowing Lemma mentioned above uses Poincaré Recurrence Theorem
- In conclusion, there exist trajectories of Φ_t^ε that 'diffuse' along the chain of cylinders $\mathcal{C}_1, \dots, \mathcal{C}_k$ and visit a δ-neighborhood of each essential torus $\mathcal{T}$
- No control on diffusion time

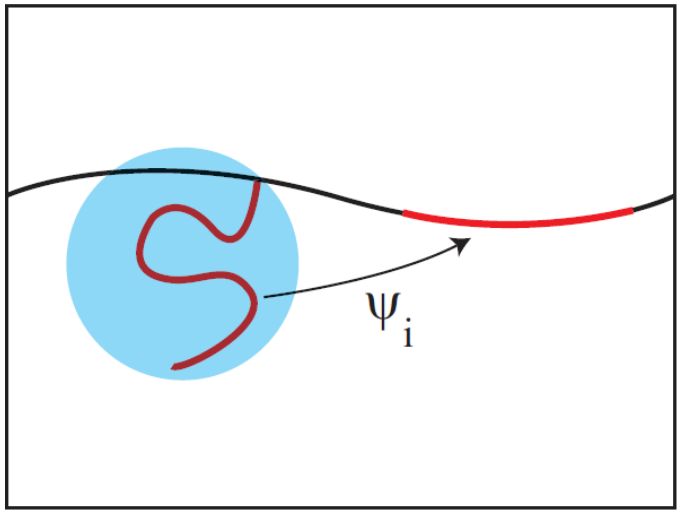
Sketch of the proof of the main result

- What do we mean by a 'special twist map'?
  - each $\Gamma \in \mathrm{Ess}(\phi_i)$ has irrational rotation number
  - each $\Gamma \in \mathrm{Ess}(\phi_i)$ is either the upper bound of a Birkhoff Zone of Instability (BZI) or is a limit from below, in the Hausdorff topology, of essential circles $\Gamma_n \prec \Gamma$ in $\mathrm{Ess}(\phi_i)$
  - these conditions are generic (Herman,1983)



Sketch of the proof of the main result

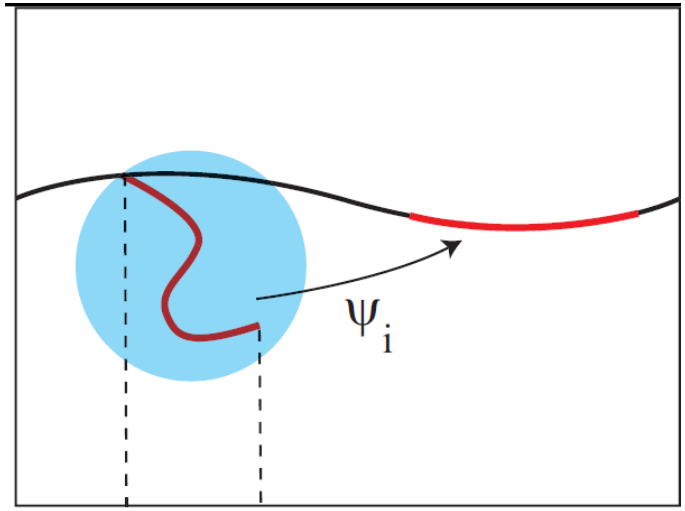
- What do we mean by the inner and the outer dynamics being in 'general position'?
  - Assumptions on the outer dynamics
    - Splitting property: for each $\Gamma \in \mathrm{Ess}(\phi_i)$ which is a limit from below of Γ_n's, there exists a splitting arc $A \subset \Gamma^-$ (= the region in A_i below Γ) with $\psi_i(A) \subseteq \Gamma$
    - Right (left) triangle property: for each Γ which is the upper bound of a BZI there exists a right (left) triangle $T \subseteq \Gamma^-$ with $\psi_i(T) \subseteq \Gamma^+$ (= the region in A_i above Γ)



A splitting arc

Sketch of the proof of the main result

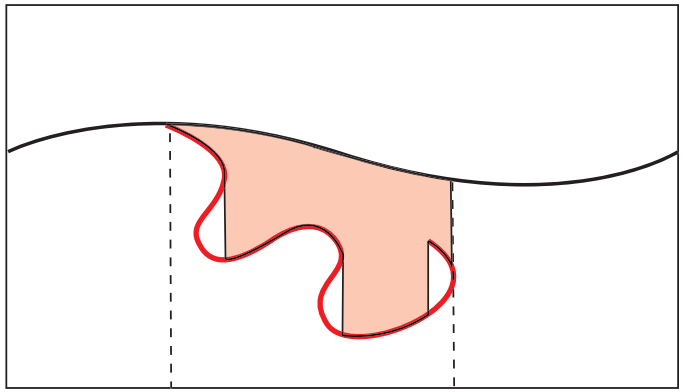
- What do we mean by the inner and the outer dynamics being in 'general position'?
  - Assumptions on the outer dynamics
    - Splitting property: for each $\Gamma \in \mathrm{Ess}(\phi_i)$ which is a limit from below of Γ_n's, there exists a splitting arc $A \subset \Gamma^-$ (= the region in A_i below Γ) with $\psi_i(A) \subseteq \Gamma$
    - Right (left) triangle property: for each Γ which is the upper bound of a BZI there exists a right (left) triangle $T \subseteq \Gamma^-$ with $\psi_i(T) \subseteq \Gamma^+$ (= the region in A_i above Γ)



A right splitting arc

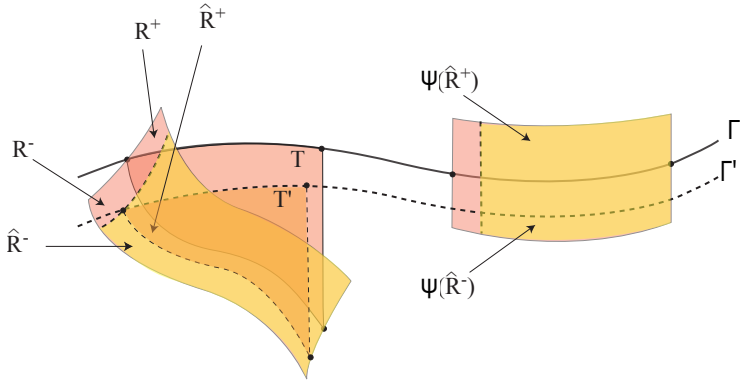
Sketch of the proof of the main result

- What do we mean by the inner and the outer dynamics being in 'general position'?
  - Assumptions on the outer dynamics
    - Splitting property: for each $\Gamma \in \mathrm{Ess}(\phi_i)$ which is a limit from below of Γ_n's, there exists a splitting arc $A \subset \Gamma^-$ (= the region in A_i below Γ) with $\psi_i(A) \subseteq \Gamma$
    - Right (left) triangle property: for each Γ which is the upper bound of a BZI there exists a right (left) triangle $T \subseteq \Gamma^-$ with $\psi_i(T) \subseteq \Gamma^+$ (= the region in A_i above Γ)



Sketch of the proof of the main result

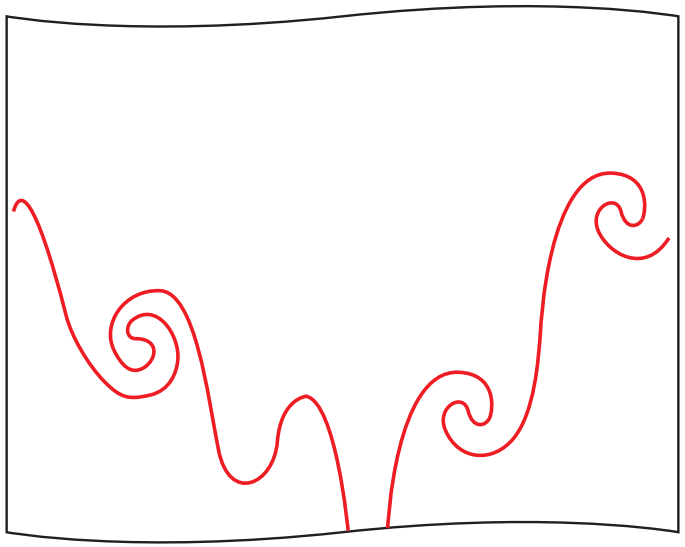
- Constructive version:
  - apply the Birkhoff procedure to ϕ until you get an essential circle Γ
  - apply the scattering map ψ to Γ and apply again the Birkhoff procedure to ϕ
  - key idea – essential circles obtained via this procedure cannot accumulate – due to the right (left) triangle property
  - Lemma: If T is a right (left) triangle for Γ and $\Gamma' \prec \Gamma$ is sufficiently close to Γ, then there exists a right (left) triangle T' for Γ' such that $\psi(T') \cap \Gamma^+ \neq \emptyset$



Sketch of the proof of the main result

How to use 'left triangles' instead of 'right triangles'

- In the above constructions take negative iterates ϕ^{-n}, $n > 0$ of the twist map ϕ, rather than positive ones
- Key ingredient: positive iterates of a vertical line under ϕ — right tilted curves; negative iterates of a vertical line under ϕ^{-1} — left tilted curves
- The orbit of $N_\bullet$ under the polysystem $\{\phi, \psi\}$ is dense in the orbit of $N_\bullet$ under the polysystem $\{\phi, \phi^{-1}, \psi\}$ – use Poincaré Recurrence Theorem



Sketch of the proof of the main result

Shadowing Lemma for NHIM's (M.G.,de la Llave,Seara,2014)

Assume that $f : M \to M$ is a smooth map, $\mathcal{C} \subseteq M$ is a NHIM, ψ is a scattering map on some open subset of $\mathcal{C}$, and $\phi = f_{|\mathcal{C}}$.
Then, for every $\delta > 0$ there exist $n^* \in \mathbb{N}$, and $m_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, s.t., for every pseudo-orbit $\{y_i\}_{i\geq 0}$ in $\mathcal{C}$ of the form

$$y_{i+1} = \phi^{m_i} \circ \psi^{\Gamma} \circ \phi^{n_i}(y_i),$$

with $n_i \geq n^*$ and $m_i \geq m_i^*(n_0, \ldots, n_{i-1}, n_i, m_0, \ldots, m_{i-1})$, there exists an orbit $\{z_i\}_{i\geq 0}$ of f in M such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i), \text{ and } d(z_i, y_i) < \delta.$$

- Remark: No assumption on the 'inner dynamics' given by ϕ