Ergodicity of the Liouville system implies the Chowla conjecture

Nikos Frantzikinakis

University of Crete, Greece

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The Liouville function

Definition

Liouville function: If $n = p_1^{a_1} \cdots p_k^{a_k}$, then $\lambda(n) = (-1)^{a_1 + \cdots + a_k}$.

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- Its signs appear to have "random type behavior". Based on this several well known conjectures have been formulated.
- Natural belief: All sign patterns appear equally frequently in the range of λ. Hence, all size k patterns occur with frequency $\frac{1}{2^k}$.
- But... Not even known that all size 4 patterns occur infinitely often!

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Sign patterns in the range of λ

Some known results about sign patterns taken by λ :

- Size 1 patterns: Both occur with density $\frac{1}{2}$ (PNT).
- Size 2 patterns: All four occur infinitely often (Harman, Pintz, Wolke 85), positive lower density (Matomäki, Radziwiłł, 2015), logarithmic density $\frac{1}{4}$ (Tao 2015).
- Size 3 patterns: All eight occur infinitely often (Hilderbrand 1986), positive lower density (Matomäki, Radziwiłł, Tao, 2015).
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Chowla conjecture

Chowla Conjecture (1965)

If $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$ are distinct, then

$$\lim_{M\to\infty} \mathbb{E}_{m\in[M]} \lambda(m+n_1)\cdots\lambda(m+n_\ell) = 0.$$

- $\ell = 1$ (PNT): $\lim_{M\to\infty} \mathbb{E}_{m\in[M]} \lambda(m) = 0.$
- $\ell = 2$ (Tao 2015): Proof for logarithmic averages. For every $n \in \mathbb{N}$

$$\lim_{M\to\infty} \frac{1}{\log M} \sum_{m=1}^{M} \frac{1}{m} \lambda(m)\,\lambda(m+n) = 0.$$

- Open for $\ell \geq 3$ even for logarithmic averages for all choices of distinct $n_1, \ldots, n_\ell \in \mathbb{N}$.
- The Chowla conjecture implies a conjecture of Sarnak on Möbius disjointness.

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A simplifying assumption

For clarity purposes and in order to ease notation we assume

Simplifying assumption

The Liouville function **admits correlations,** meaning, the limit

$$\lim_{M\to\infty} \mathbb{E}_{m\in[M]}\lambda(m+n_1)\cdots\lambda(m+n_\ell)$$

exists for every $\ell \in \mathbb{N}$ and $n_1, \ldots, n_\ell \in \mathbb{N}$.

In the general case, we work with any subsequence of intervals $([M_k])_{k\in\mathbb{N}}$ along which λ **admits correlations.** Then we get Chowla-type results for **logarithmic averages** along $([M_k])_{k\in\mathbb{N}}$.

Notation

- $\mathbb{E}_{n\in\mathbb{N}}a(n) = \lim_{N\to\infty} \mathbb{E}_{n\in[N]}a(n)$ if the limit exists.
- $\mathbb{E}_{n\in\mathbb{N}}a(n) = \limsup_{N\to\infty} \mathbb{E}_{n\in[N]}a(n)$ if $a(n) \geq 0$.

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Ergodic reinterpretation of Chowla conjecture

Furstenberg Correspondence Principle

If $a \in \ell^\infty(\mathbb{N})$ admits correlations, then there exist a measure preserving system $(X, \mathcal{X}, \mu, T)$ and a function $f \in L^\infty(\mu)$ such that

$$\int T^{n_1} f \cdots T^{n_\ell} f \, d\mu = \mathbb{E}_{m\in\mathbb{N}} a(m+n_1)\cdots a(m+n_\ell),$$

for all $n_1, \ldots, n_\ell \in \mathbb{Z}$. If $a = \lambda$, we call the system **Liouville system**.

- $X = D^{\mathbb{Z}}$, $(Tx)(k) = x(k+1)$, $f(x) = x(0)$, only μ varies.
- Chowla conjecture $\Rightarrow$ Liouville system is a Bernoulli system. Not known if the Liouville system is weakly mixing or even ergodic.
- Ergodic point of view also used by el Abdalaoui, Avdeeva, Cellarosi, Kułaga-Przymus, Lemańczyk, de la Rue, Sarnak, Sinai, Weiss etc, to study properties of systems related to the Möbius and Liouville function, square-free and B-free integers.

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Ergodicity implies the Chowla conjecture

Main Result

If the Liouville system is ergodic, then it is Bernoulli and the Chowla conjecture holds.

Equivalently, if the Liouville function is generic for an ergodic measure, then this measure is Bernoulli and the Chowla conjecture holds.

Main Result (not assuming existence of correlations)

If the Liouville function admits correlations for logarithmic averages along $([M_k])$ and the corresponding system is ergodic, then the Chowla (and the Sarnak) conjecture holds for **logarithmic averages along** $([M_k])$.

Averaging operation used: $\frac{1}{\log M_k} \sum_{m \in [M_k]} \frac{1}{m} \cdots$.

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Main steps in the proof

The proof contains three main ingredients:

1. Tao (2015): Local uniformity of the Liouville function implies the Chowla conjecture.
2. An inverse theorem for local uniformity seminorms of ergodic sequences.
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Local uniformity seminorms

Definition (Host, Kra 2009)

If $a \in \ell^\infty(\mathbb{N})$ admits correlations, we let $(S_r a)(n) := a(n+r)$ and

$$\|a\|^2_{U^1(\mathbb{N})} := \mathbb{E}_{r\in\mathbb{N}}\mathbb{E}_{n\in\mathbb{N}}a(n+r)\cdot\overline{a(n)},$$

$$\|a\|^{2^{s+1}}_{U^{s+1}(\mathbb{N})} := \mathbb{E}_{r\in\mathbb{N}}\left\|S_r a\cdot\overline{a}\right\|^{2^s}_{U^s(\mathbb{N})}.$$

- $\|a\|^4_{U^2(\mathbb{N})} = \mathbb{E}_{r,s\in\mathbb{N}}\left(\mathbb{E}_{n\in\mathbb{N}}\, a(n)\cdot\overline{a(n+r)}\cdot\overline{a(n+s)}\cdot a(n+r+s)\right).$
- All limits can be shown to exist (using the ergodic reinterpretation).
- If $(a(n))_{n\in\mathbb{N}}$ is ergodic, then $\|a\|_{U^1(\mathbb{N})} = \left|\mathbb{E}_{n\in\mathbb{N}}a(n)\right|$ and

$$\|a\|^4_{U^2(\mathbb{N})} = \mathbb{E}_{r\in\mathbb{N}}\left|\mathbb{E}_{n\in\mathbb{N}}a(n+r)\cdot\overline{a(n)}\right|^2.$$

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- Ergodic reinterpretation: If $(X, \mathcal{X}, \mu, T)$ is the system and f is the function associated to $(a(n))_{n\in\mathbb{N}}$, then $\|a\|_{U^s(\mathbb{N})} = \|\!|f|\!\|_s$ where $\|\!|\cdot|\!\|_s$ are the Host-Kra seminorms:

$$\|\!|f|\!\|_1^2 = \mathbb{E}_{r\in\mathbb{N}}\int T^r f\cdot\overline{f}\, d\mu, \quad \|\!|f|\!\|_{s+1}^{2^{s+1}} := \mathbb{E}_{r\in\mathbb{N}}\|\!|T^r f\cdot\overline{f}|\!\|_s^{2^s}.$$

Local uniformity implies Chowla

Theorem (Tao 2015)

$\|\lambda\|_{U^s(\mathbb{N})} = 0$ *for every* $s \in \mathbb{N} \iff$ *The Chowla conjecture is satisfied.*

Hence, our main result follows from:

Theorem ($U^s(\mathbb{N})$-uniformity for λ)

If the Liouville system is ergodic, then $\|\lambda\|_{U^s(\mathbb{N})} = 0$ *for every* $s \in \mathbb{N}$.

- Gowers uniformity is known for λ (Green, Tao, Ziegler 2012), but this is a much weaker condition than local uniformity.
- $\|\lambda\|_{U^1(\mathbb{N})} = 0 \iff \lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} |\mathbb{E}_{n\in[m,m+N]} \lambda(n)| = 0$ which is known by Matomäki, Radziwiłł (2015).
- $\|\lambda\|_{U^2(\mathbb{N})} = 0$ is an open problem. It is equivalent to

$$\lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} \sup_t |\mathbb{E}_{n\in[m,m+N]} \lambda(n)\, e(nt)| = 0.$$

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Local uniformity implies Chowla

Theorem (Tao 2015)

$\|\lambda\|_{U^s(\mathbb{N})} = 0$ *for every* $s \in \mathbb{N} \iff$ *The Chowla conjecture is satisfied.*

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Nilsequences

A class of sequences that obstructs (local) uniformity:

Examples (of Nilsequences)

- $\psi(n) = e(n^s\alpha)$ or $\psi(n) = e(p(n))$ with $\deg(p) = s$.
- But also $\psi(n) = e([n\alpha]n^{s-1}\beta)$ or $\psi(n) = e([[n\alpha]n\beta]n^{s-2}\gamma)$.

Definition (Nilsequences)

- $X = G/\Gamma$ is an s-step nilmanifold, $b \in G$, $\Psi \in C(X)$, then $\psi(n) = \Psi(b^n \cdot \Gamma)$ is an **s-step nilsequence.**
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Step 1: An inverse theorem for ergodic sequences

Theorem (Inverse theorem for $U^s(\mathbb{N})$-seminorms)

Let $a \in \ell^\infty(\mathbb{N})$ be an ergodic sequence. Then $\|a\|_{U^{s+1}(\mathbb{N})} = 0$ if and only if for every s-step nilsequence ϕ and every $(s-1)$-step nilmanifold Y

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Proof of inverse theorem for $U^2(\mathbb{N})$-seminorms

Suppose that $\|a\|_{U^2(\mathbb{N})} > 0$.

1. **Ergodicity** implies $\|a\|_{U^2(\mathbb{N})}^4 = \mathbb{E}_{r\in\mathbb{N}}\big|\mathbb{E}_{n\in\mathbb{N}} a(n)\cdot\overline{a(n+r)}\big|^2$, hence

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The inverse condition for the Liouville function

Theorem (Orthogonality of λ with nilsequences)

Suppose that the Liouville system is ergodic. Then for every $s \in \mathbb{N}$, for every s-step nilsequence ϕ and every $(s-1)$-step nilmanifold Y

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Proof by induction on $s \in \mathbb{N}$. Schematically

MRT15 $\Rightarrow$		$\lambda \perp$ 1-step nil	($\mathbf{s} = \mathbf{1}$ case)
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Induction step (Reduction to a dynamical property)

Using an orthogonality criterion of Kátai (86) we reduce matters to showing the following statement of purely dynamical context:

Theorem (Orthogonality of irrational nilsequences)

Under an irrationality condition on the s-step nilsequence $(\Phi(b^n \cdot \Gamma))$, if Y is an $(s-1)$-step nilmanifold, $p, q \in \mathbb{N}$ with $p \neq q$, $(I_N)_{N\in\mathbb{N}}$ intervals with $|I_N| \to \infty$, then

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Follows by studying the structure of the nilmanifold

$$Y = \overline{\{(b^{pn}\cdot\Gamma, b^{qn}\cdot\Gamma), n\in\mathbb{N}\}}.$$

Key property: $Y = H/\Delta$ where $(u^{p^s}, u^{q^s}) \in H$ for every $u \in G_s$.

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Open problems

Problem (Ergodicity of the Liouville system)

Suppose that the Liouville function admits correlations. Show that the induced system **is ergodic.**

- A variant for logarithmic averages would imply the Chowla (and Sarnak) conjecture for logarithmic averages.

Easier Problem (The Liouville system is not a Kronecker system)

Suppose that the Liouville function admits correlations. Show that the induced system **is not a (non-ergodic) mixture of circle rotations.**

- It is not even clear how to exclude the possibility that λ can be approximated in density by "local 1-step nilsequences".

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Sketch of proof of the dynamical property

- Apply van der Corput $s-1$ times reduces matters to showing: $\|\Phi \otimes \overline{\Phi}\|_{s,Y} = 0$, for the rotation by (b^p, b^q) acting on

$$Y = \overline{\{(b^{pn} \cdot e_X, b^{qn} \cdot e_X), n \in \mathbb{N}\}}.$$

- Key observation: $Y = H/\Delta$ where $\Gamma \times \Gamma \subset H$ and

$$(u^{p^s}, u^{q^s}) \in H_s \quad \text{for every } u \in G_s.$$

  Why? We get $(g^p, g^q) \in H \cdot (G_2 \times G_2)$ for all $g \in G$ because b is totally ergodic on X. Then take iterative commutators $s-1$ times.

- It follows that $\chi \otimes \overline{\chi}$ is non-trivial on H_s, hence $\Phi \otimes \overline{\Phi}$ is a nontrivial nilcharacter of the s-step nilmanifold Y
- Hence $(\Phi \otimes \overline{\Phi}) \perp \mathcal{Z}_{s-1}(Y)$ (by Host-Kra 05) $\Longrightarrow \|\Phi \otimes \overline{\Phi}\|_{s,Y} = 0$.

Inverse theorem (Sketch of proof for $U^3(\mathbb{N})$)

Suppose that $\|a\|_{U^3(\mathbb{N})} > 0$.

1. **Ergodicity** implies

$$\mathbb{E}_{r,s\in\mathbb{N}}\big(\mathbb{E}_{n\in\mathbb{N}} a(n+r+s)\cdot\overline{a(n+r)}\cdot\overline{a(n+s)}\cdot a(n)\cdot A(r,s)\big) > 0,$$

$$A(r,s) := \mathbb{E}_{n\in\mathbb{N}} a(n+r+s)\cdot\overline{a(n+r)}\cdot\overline{a(n+s)}\cdot a(n).$$

2. Using an ergodic structure theorem of Host and Kra (05) (ergodic theory is crucial at this point) we get

$$A(r,s) = \Phi(r,s) + E(r,s),$$

such that

- $\Phi(r,s) = \mathbb{E}_{n\in\mathbb{N}}\, \phi(n+r+s)\,\overline{\phi(n+r)}\,\overline{\phi(n+s)}\,\phi(n)$ where ϕ **is a** 2 **step nilsequence;**
- $\mathbb{E}_{r,s\in\mathbb{N}}|E(r,s)| = 0.$

Inverse theorem (Sketch of proof for $U^3(\mathbb{N})$)

3. For convenience say $\Phi(r,s) = e(rs\alpha)$. Then for

$$b(n) := a(n) \cdot e(n^2\alpha)$$

we have

$$\|b\|_{U^2(\mathbb{N})} = \mathbb{E}_{r,s\in\mathbb{N}}\big(\mathbb{E}_{n\in\mathbb{N}} b(n+r+s)\cdot\overline{b(n+r)}\cdot\overline{b(n+s)}\cdot b(n)\big) > 0,$$

4. Using this and a finitistic decomposition result of Green and Tao we deduce

$$\limsup_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} \sup_t |\mathbb{E}_{n\in[m,m+N]} a(n)\, e(n^2\alpha)\, e(nt)| > 0.$$

Some facts about the Liouville system

- (Matomäki, Radziwiłł 2015): f **is orthogonal to the invariant factor** of the Liouville system since

$$\mathbb{E}_{n\in\mathbb{N}} \int f \cdot T^n f \, d\mu_\lambda = 0 \Longleftrightarrow \lim_{N\to\infty} \mathbb{E}_{m\in\mathbb{N}} |\mathbb{E}_{n\in[m,m+N]} \lambda(n)| = 0.$$

- (Matomäki, Radziwiłł, Tao 2015): f **is orthogonal to the Kronecker factor** of the Liouville system. Follows from

$$\mathbb{E}_{n\in\mathbb{N}} \Big| \int f \cdot T^n f \, d\mu_\lambda \Big| = 0 \Longleftrightarrow \mathbb{E}_{n\in\mathbb{N}} \Big| \mathbb{E}_{m\in\mathbb{N}} \lambda(m) \cdot \lambda(m+n) \Big| = 0.$$

- **It is not known if f is orthogonal to $\mathcal{Z}_1(\mu_\lambda)$.** If $\mu_\lambda = \int \mu_x \, d\mu_\lambda$ is the ergodic decomposition of μ_λ, then $\mathbb{E}_{n\in\mathbb{N}} \int | \int f \cdot T^n f \, d\mu_x|^2 \, d\mu_\lambda = 0 \Longleftrightarrow$

$$\mathbb{E}_{n\in\mathbb{N}} \mathbb{E}_{r\in\mathbb{N}} \mathbb{E}_{m\in\mathbb{N}} \, \lambda(m) \cdot \lambda(m+n) \cdot \lambda(m+r) \cdot \lambda(m+n+r) = 0,$$

and this is equivalent to $\|\lambda\|_{U^2(\mathbb{N})} = 0$ (to be defined shortly).