A Study of Stochastic Resonance as a Periodic Random Dynamical System.

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What is the stochastic resonance?

The concept of stochastic resonance indicates a class of phenomena in nonlinear systems where a weak 'signal' can be amplified and optimized by the presence of noise.

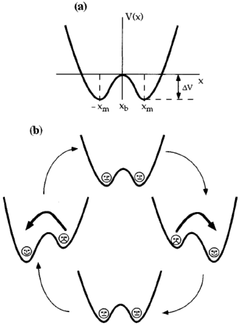
It was originally introduced to explain the periodic recursion of glaciations: the periodic effect of the changes in time of the eccentricity of the earth's orbit around the sun is amplified by environmental noise given by atmosferic fluctuations.

After that, a number of other phenomena in physics and biology were explained by the role played by the noise in the amplification of a signal.

- R. Benzi, G. Parisi and A. Sutera, *The mechanism of stochastic resonance* , Journal of Physics A **14** (1981)
- R. Benzi, G. Parisi, A. Sutera and A. Vulpiani, *A theory of stochastic resonance in climatic change*, SIAM Journal on Applied Mathematics **83** (1983), 565–578.
- C. Nicolis, *Solar variability and stochastic effects on climate*, Sol. Phys. 74, 473-478 (1981).

A model for stochastic resonance

A simple model for stochastic resonance is given by a damped particle in a periodically oscillating double-well potential in the presence of noise: when applying a periodic forcing, the double-well potential is tilted asymmetrically up and down.



L. Gammaitoni et al., *Stochastic resonance*, Reviews of Modern Physics **70**(1998)

A model for stochastic resonance

- For weak periodic forcing, the noise strenght can be tuned so that noise-induced hoppings between the wells become synchronised with the periodic forcing: the average waiting time between two inter-well transitions is comparable with the period of the forcing.
- Outside the resonant range of parameters, for increasing noise strength, the periodicity is lost and the hopping becomes increasingly random.

One-dim approximation and SDE

The phenomenon appears also in the one-dimensional approximation of this model, which describes the dynamics of an overdamped particle:

$$dx = (\alpha x - \beta x^3)dt + \mathcal{A}\cos\nu t\ dt + \sigma dW_t \quad \alpha, \beta, \sigma > 0, \quad x \in \mathbb{R} \quad (1)$$

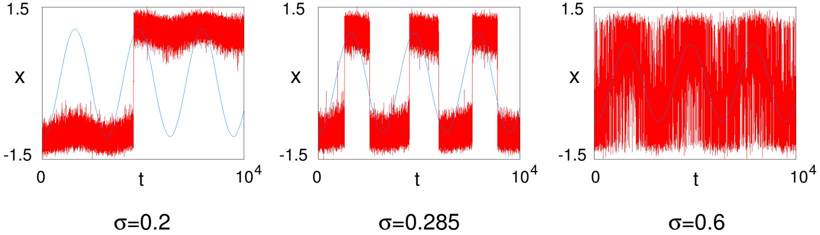


Figure 1: Paths for the SDE at increasing value of noise

What can we say about the stochastic resonance from the point of view of dynamical systems?

AMC, J.S.W. Lamb, M. Ramussen, Y. Sato

A Random Dynamical Systems Perspective on Stochastic Resonance

to appear in *Nonlinearity*

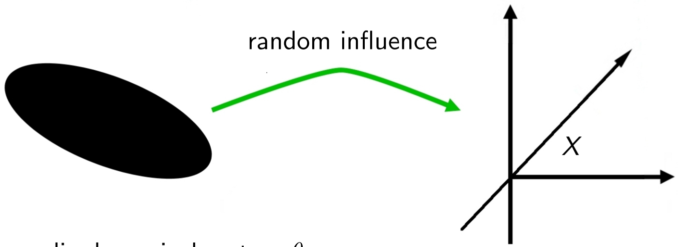
Outline

- Framework for nonautonomous random dynamical systems.
- Existence of a global attractor for a class of periodic RDSs including the SR.
- The attractor is a random periodic orbit associated to a families of invariant measures.
- Definition of an indicator for the SR.

Random dynamical systems

A random dynamical system is the combination of two systems: a base flow θ in a probability space, and a flow given by a cocycle in a phase space X. The evolution in the phase space may depend on maps, or on a system of differential equations randomly chosen; the underlying system describes the evolution of noise.

model for the noise — model for the dynamics



ergodic dynamical system θ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

cocycle over θ on X metric space

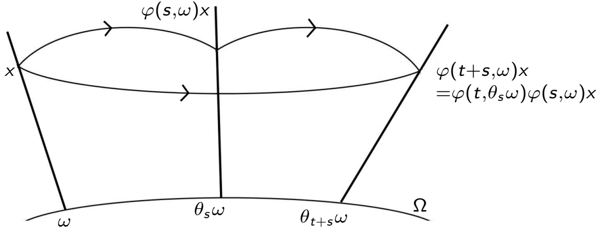
Model for the noise

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
a time set $\mathbb{T}$ ($\mathbb{R}, \mathbb{R}^+$ or $\mathbb{Z}$);
a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F})$-measurable function $\theta : \mathbb{T} \times \Omega \to \Omega$ is an ergodic dynamical system if:

(i) Initial value condition: $\theta_0 \omega = \omega$.
(ii) Group property: $\theta_{t+s}\omega = \theta_t(\theta_s \omega)$.
(iii) Invariance: $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$.
(iv) Ergodicity: $\theta_t A = A \Longrightarrow \mathbb{P}(A) \in \{0, 1\}$.

Cocycle in the "autonomous" case

In the standard definition, i.e. in absence of nonautonomous deterministic forcing, the model of dynamics is a cocycle over the ergodic dynamical system θ and time.



The nonautonomous case: same model for noise, different cocycle.

For random systems with a nonautonomous deterministic component, the dynamics depends on the initial time: we keep the same model for the evolution of noise and we add to the cocycle a variable accounting for the initial time.

Let $X = \mathbb{R}^d$. Then

$\Phi : \mathbb{T} \times \mathbb{T} \times \Omega \times X \to X$, $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable, is a mapping with the cocycle properties:

(i) $\Phi(0, \tau, \omega, x) = x$ for all $\tau \in \mathbb{T}$, $\omega \in \Omega$ and $x \in X$,

(ii) $\Phi(t + s, \tau, \omega, x) = \Phi(t, \tau + s, \theta_s \omega, \Phi(s, \tau, \omega, x))$ for all $t, s, \tau \in \mathbb{T}$, $\omega \in \Omega$ and $x \in X$.

(We will use also the notation $\Phi(t, \tau, \omega)x$ for $\Phi(t, \tau, \omega, x)$).

Periodic random dynamical system

We say that the system is periodic if there exists a number $T > 0$ such that

$$\Phi(t, \tau + T, \omega, x) = \Phi(t, \tau, \omega, x) \text{ for all } t, \tau \in \mathbb{T},\ \omega \in \Omega,\ x \in X$$

Example: discrete-time case

Consider a metric space X and four homeomorphisms $h^i_j : X \to X$, $i, j = 0, 1$.

We want to study the random dynamics if h^i_j is used with probability p_j at either even times ($i = 0$) or odd times ($i = 1$).

(i) The ergodic dynamical system θ is given by:
$\Omega := \{\omega = (\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots) : \omega_i \in \{0,1\}\}$.
$\mathcal{F} = \sigma$(cylinder sets).
$\mathbb{P}(I_{x_1,\ldots,x_n}) := \prod_{i=1}^n p_{x_i}$ defined on cylinder sets $I_{x_1,\ldots,x_n}$, $x_i \in \{0,1\}$, then extended to $\mathcal{F}$.
θ left shift: $(\theta(\omega))_i = \omega_{i+1}$.

(ii) Cocycle:
$\varphi(1, m, \omega)x := (h^{m \mod 2}_{\omega_0})(x)$ for $m \in \mathbb{N}$.

The system is periodic and the cocycle property is proved by:

$$\varphi(n-1, m+1, \theta_1\omega)\varphi(1, m, \omega)x =$$
$$= (h^{m+n-1 \mod 2}_{\omega_{n-1}} \circ \cdots \circ h^{m+1 \mod 2}_{\omega_1}) \circ (h_{\omega_0}{}^{m \mod 2}(x)) = \varphi(n, m, \omega)x\,.$$

Continous case: nonautonomous RDSs generated by a SDE

Given a 1-dim SDE

$$\mathrm{d}x = f(x,t)\mathrm{d}t + \sigma \mathrm{d}W_t, \sigma > 0, \quad x \in \mathbb{R}$$

$(W_t)_{t\in\mathbb{R}}$ is a Wiener process.

- The model for the noise is:
  - $\Omega := C_0(\mathbb{R},\mathbb{R}) := \{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\}$ equipped with the Borel σ-algebra $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$
  - $\mathbb{P}$ is the Wiener probability measure on $(\Omega,\mathcal{F})$.
  - The evolution of noise is described by the Wiener shift $\theta : \mathbb{R} \times \Omega \to \Omega$, defined by $\theta(t,\omega(\cdot)) := \omega(\cdot + t) - \omega(t)$.
- Let $\xi(\cdot,\tau,\omega,x)$ be the stochastic flow of the SDE, for initial time τ and initial condition $x \in \mathbb{R}$; the cocycle is defined by

$$\Phi(t,\tau,\theta_\tau\omega,x) := \xi(t+\tau,\tau,\omega,x)$$

Basic concepts in the nonautonomous case

- Invariant random sets
- Global random attractors
- Absorbing sets

Nonautonomous invariant random sets

- We call nonautonomous random set a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X))$-measurable set $M \subset \mathbb{T} \times \Omega \times X$.
- $M(\tau,\omega) := \{x \in X : (\tau,\omega,x) \in M\}$ is called the (τ,ω)-*fiber* of M.
- If each fiber is *closed (compact or bounded...)*, then M is called *closed* (*compact*, or *bounded*).
- If each fiber is constant $M(\tau,\omega) \equiv B \subset X$, M is called deterministic set

A random set M is invariant with respect to the RDS iff.

$$\Phi(t,\tau,\omega)M(\tau,\omega) = M(\tau + t, \theta_t\omega)$$

for all $t, \tau, \in \mathbb{T}$ and for almost all $\omega \in \Omega$.

It is always possible to construct one.

Random Periodic Orbit

- A nonautonomous invariant random set M whose fibers are points is called random periodic orbit if there exists a positive time T such that

  $$\Phi(T, \tau, \omega) M(\tau, \omega) = M(\tau, \theta_T \omega)$$

  for all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$

  NB Also: $M(\tau + T, \omega) = M(\tau, \omega)$. Apply invariance and periodicity to $M(\tau, \theta_{-T}\omega)$...

Nonautonomous global random attractors

We are interested in global nonautonomous random attractors for the family of all deterministic bounded sets (but the notion can be generalised).

Nonautonomous global random attractors

A nonautonomous random set A is a global attractor if

- A is invariant;
- A attracts in the pullback sense every deterministic bounded set, i.e.

$$\lim_{t\to\infty} \mathrm{dist}\left(\Phi(t,\tau-t,\theta_{-t}\omega)B, A(\tau,\omega)\right) = 0$$

for all $\tau \in \mathbb{T}$, for almost all $\omega \in \Omega$ and for all bounded $B \subset X$.[1]

[1] $\mathrm{dist}(D_1, D_2) := \sup_{x\in D_1} \inf_{y\in D_2} d(x,y)$ is the Hausdorff semi-distance of two sets $D_1, D_2 \subset X$.

Example: globally attracting random periodic orbit.

- For any bounded set $B \subset \mathbb{R}$,

$$\lim_{t \to \infty} \Phi(t, \tau - t, \theta_{-t}\omega)B = A(\tau, \omega)$$

for all τ and almost all ω, $A(\tau, \omega)$ is a point;

- $A(\tau, \cdot)$ is a random variable that evolves under the stochastic flow as

$$\Phi(t, \tau, \omega)A(\tau, \omega) = A(\tau + t, \theta_t \omega)$$

and

for a $T > 0$,

$$A(\tau + T, \omega) = A(\tau, \omega)$$

($T = \frac{2\pi}{\nu}$ for the SR).

Existence of global random attractors

Absorbing set

A compact nonautonomous random set K is absorbing if for all deterministic bounded sets B, all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$, there exists a time $T(B, \tau, \omega) > 0$ s.t.

$$\Phi(t, \tau - t, \theta_{-t}\omega)B \subset K(\tau, \omega)$$

.

The existence of an absorbing set is a sufficient condition for the existence of a global nonautonomous random attractor

Theorem 1

Suppose that there exists an absorbing set K. Then there exists a global nonautonomous random attractor A, given by the omega-limit set of K:

$$A(\tau,\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \Phi(t,\tau-t,\theta_{-t}\omega)K(\tau-t,\theta_{-t}\omega)}$$

for all $\tau \in \mathbb{T}$ and almost all $\omega \in \Omega$.

- A is minimal: if there is another global nonautonomous random attractor $\tilde{A}$, then $A(\tau,\omega) \subset \tilde{A}(\tau,\omega)$ for all $\tau \in \mathbb{T}$ and almost all $\omega \in \Omega$.
- If X is connected, then the fibers of A are connected.

The nonautonomous RDS generated by the stochastic resonance has a global random attractor

Existence of a global attractor for the stochastic resonance

The SDE for the stochastic resonance

$$\mathrm{d}x = \underbrace{\left(\alpha x - \beta x^3 + \mathcal{A}\cos\nu t\ \right)}_{f(x,t)}\mathrm{d}t + \sigma\mathrm{d}W_t \quad \alpha,\beta,\sigma > 0, \quad x \in \mathbb{R}$$

generates a nonautonomous RDS (θ, Φ), the cocycle Φ is given by the stochastic flow X_t.

- Φ has a nonautonomous absorbing set, given by $B\left(O_\tau(\omega), R(\tau,\omega)\right)$, "ball" of radius $R(\tau,\omega)$ centered in the Ornstein-Uhlenbeck process $O_t(\omega)$.
- Then Φ has a nonautonomous global random attractor, which is a periodic, compact and connected nonautonomous random set: each fiber $A(\tau,\omega)$ is an interval in $\mathbb{R}$.

Proposition

A nonautonomous random dynamical system generated by the SDE

$$\mathrm{d}x = f(x,t)\mathrm{d}t + \sigma \mathrm{d}W_t \quad x \in \mathbb{R} \tag{2}$$

where f satisfies:

i *Dissipativity condition.* There exist constants $L_1, L_2 \geq 0$ such that $(x_1 - x_2)\big(f(x_1,t) - f(x_2,t)\big) \leq L_1 - L_2 \left|x_1 - x_2\right|^2$.

ii *Integrability condition.* $\exists C_0 > 0$ s.t. $\int_{-\infty}^{t} e^{c\,r} \big|f(u(r),r)\big|^2 dr < \infty$, for all $0 < c < C_0$ and u continous in $\mathbb{R}$ with sub-exponential growth.

has a global nonautonomous random attractor.

Proof

The main idea here is to transform the SDE (2) into a random ordinary differential equation

Given the SDE in $\mathbb{R}$ $\mathrm{d}y = -y\mathrm{d}t + \sigma \mathrm{d}W_t$, with the pathwise solution $Y(t, \tau, \omega, y_\tau) = y_\tau e^{-t} + \sigma e^{-t} \int_\tau^t e^r dW_r$; the pullback limit Is the Ornstein-Uhlenbeck process:

$$O_t(\omega) = \sigma e^{-t} \int_{-\infty}^{t} e^r dW_r$$

Let $Z_t := X_t - O_t$, X_t a pathwise solution of the SDE (2);

Z_t satisfies the random ordinary differential equation (RODE)

$$\frac{dz}{dt} = f(z + O_t(\omega), t) + O_t(\omega)$$

associated to the cocycle $\Psi(s, \tau, \omega)z =_{def.} Z(\tau + s, \tau, \theta_{-\tau}\omega, z)$
for all $t, \tau, \in \mathbb{R}, \omega \in \Omega, z \in \mathbb{R}$.

The proof is based on differential inequalities for $Z_t = Z(t, \tau, \omega, z)$ solution of the RODE.

The global random attractor is a periodic orbit.

- The main idea for proving the triviality of the attractor is to prove properties of invariant measures: in particular, we prove a one-to-one correspondence between 'periodic measures' for RDSs and stationary periodic measures for the correspondent Markov semigroups.
- If there is a unique 'periodic invariant measure' then each fiber of the attractor is a point

Invariant nonautonomous and invariant periodic measures.

The skew product for the nonautonomous RDS is the map
$\Theta : \mathbb{T} \times \mathbb{T} \times \Omega \times X \mapsto \mathbb{T} \times X \times \Omega$

$$(t, \tau, \omega, x) \mapsto (\tau + t, \theta_t \omega, \Phi(t, \tau, \omega)x)$$

We say that $\mu : \mathbb{T} \times \mathcal{F} \otimes \mathcal{B} \mapsto [0, 1]$ is an invariant nonautonomous measure for the NRDS if

(i) for all $\tau \in \mathbb{T}$, $\mu(\tau, \cdot)$ is a measure on $\Omega \times X$ with $\pi_\Omega \mu(\tau, \cdot) = \mathbb{P}$, where $\pi_\Omega \mu(\tau, \cdot)$ denotes the marginal on $(\Omega, \mathcal{F})$, and

(ii) for all $A \in \mathcal{F} \otimes \mathcal{B}(X)$ and $t, \tau \in \mathbb{T}$, we have

$$\mu(\Theta(t, \tau, A)) = \mu(\tau, A) .$$

We say that an invariant measure μ is invariant periodic for a nonautonomous periodic RDS if $\exists T > 0$ such that for all t

$$\mu(\tau, \cdot) = \mu(\tau + T, \cdot)$$

Disintegration of measures

We write μ_τ for $\mu(\tau, \cdot)$.

- μ_τ can be uniquely 'disintegrated' into a family $\mu_{\tau,\omega}$ of probability measures on X:

$$\mu_\tau(A) = \int_\Omega \mu_{\tau,\omega}(A_\omega)\, \mathrm{d}\mathbb{P}(\omega)\,,$$

where $A_\omega = \{x \in X : (x, \omega) \in A\}$, for all $A \in \mathcal{F} \otimes \mathcal{B}(X)$.

We can check properties of measures by properties of their disintegration:

- μ is an invariant nonautonomous measure iff.

$$\Phi(t, \tau, \omega)\mu_{\tau,\omega} = \mu_{\tau+t,\theta_t\omega}$$

for all $t, \tau, \in \mathbb{T}$ and for almost all $\omega \in \Omega$

An invariant μ is invariant periodic iff.

$$\Phi(T, \tau, \omega)\mu_{\tau,\omega} = \mu_{\tau,\theta_T\omega}$$

for all $\tau \in \mathbb{T}$ and for almost all $\omega \in \Omega$

Stationary periodic measures for the Markov semigroup

Suppose $\xi(\cdot, \tau, \omega, x)$ is the stochastic flow of the nonautonomous one-dimensional SDE .

We say that $\rho : \mathbb{T} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is a stationary nonautonomous measure for the associated non-homogeneous Markov semigroup if each $\rho(\tau, \cdot)$)or (ρ_τ) is s.t.

$$\rho_{\tau+t}(B) = \int_X Q(t, \tau, x, B) d\rho_\tau(x) \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}), \tau \in \mathbb{R}$$

where $Q(t, \tau, x, B)$ are transition probabilities for the non-homogeneous Markov semigroup:

$$Q(t, \tau, x, B) := \mathbb{P}\{\omega : \xi(t+\tau, \tau, \omega, x) \in B\} \quad \text{for all } t, \tau \in \mathbb{T},\ x \in \mathbb{R},\ B \in \mathcal{B}(\mathbb{R}).$$

We say that ρ is a stationary periodic measure if there exists a $T > 0$ such that $\rho_{\tau+T} = \rho_\tau$ for all $\tau \in \mathbb{T}$.

Theorem 2

If the stochastic differential equation (2) is T-periodic there is a one-to-one correspondence between stationary T-periodic measures for the Markov semigroup and invariant T-periodic measures for the RDS

Proof

- Define the discrete-time autonomous random dynamical system $(\tilde{\theta}, \tilde{\Phi})$ by

$$\tilde{\Phi}(n, \omega, x) := \Phi(nT, 0, \omega, x) \quad \text{for all } n \in \mathbb{Z}, \omega \in \Omega \text{ and } x \in \mathbb{R} \quad (3)$$

  and $\tilde{\theta}(\omega) := \theta_T(\omega)$ for all $\omega \in \Omega$.

- Define the transition probabilities $\tilde{Q}(x, B) := Q(T, 0, x, B)$ for the discretised system.

- There is a one-to-one correspondence $\tilde{\mu} \longleftrightarrow \tilde{\rho}$ between invariant Markov measures for (3) and stationary measures for the discrete-time Markov semigroup defined by the transition probabilities $\tilde{Q}$.

- The invariant measure $\tilde{\mu}$ can be uniquely continued to an invariant periodic measure μ for the periodic random dynamical system (θ, Φ).

- The stationary measure $\tilde{\rho}$ can be uniquely continued to a stationary periodic measure ρ for the non-homogenous Markov semigroup associated to (2).

Theorem 3

If

(i) the stochastic differential equation (2) is T-periodic,
(ii) there exists a unique family of stationary T-periodic measures for the non-homogeneous Markov semigroup and,
(iii) there exists a periodic global nonautonomous random attractor for the nonautonomous random dynamical system Φ generated by (2), then

the attractor is a random periodic orbit for Φ.

IMPORTANT: the RDS is order-preserving i.e. if $x \geq y$ then $\Phi(t, \tau, \omega, x) \geq \Phi(t, \tau, \omega, y)$, for all t, τ, for almost all ω.

Corollary: The stochastic resonance has an attracting random periodic orbit

In fact, for any choice of the parameters in the SR model the conditions are fulfilled, in fact:

- it's periodic,
- there exists a global nonautonomous random attractor and,
- there exists a unique T-periodic stationary measure for the associated non-homogeneous Markov semigroup.

Remarks

- In the case of SR, the periodic attractor exists for any choice of the parameters.
- Properties of the attractor give information on the the resonant/non resonant regimes.

An indicator for the resonance

Given $A(\tau, \omega)$ is the attracting random periodic orbit, we can define the measure $\rho_t(B) := \int \delta_{A(t,\omega)}(B)\mathrm{d}\omega$ on measurable sets $B \subset \mathbb{R}$, for $t \in \mathbb{R}$

- $\rho_t(B)$ is T-periodic, i.e. $\rho_\tau = \rho_{\tau+T}$
- The density shows a strong asymmetry in the resonant regime, progressively less marked at increasing noise.

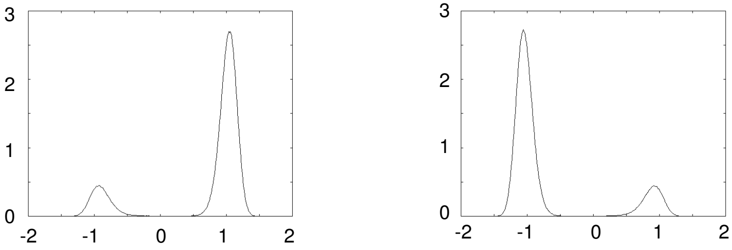


Figure 2: The Lebesgue density of ρ_t in a resonant regime at a time τ and $\tau + \frac{T}{2}$.

A very natural indicator for the resonance relates to the number of particles moving between the wells across the barrier at $x = 0$, over a time period.

$$p := p^- p^+$$

where

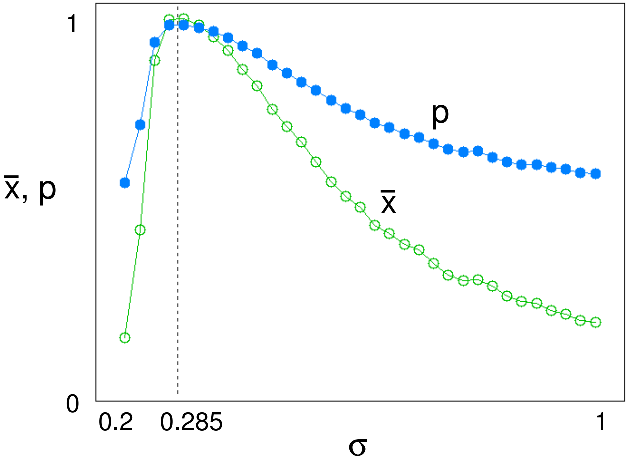
$$p^- := \frac{\max_{0\leq t<T} \rho_t((-\infty,0]) - \min_{0\leq t<T} \rho_t((-\infty,0])}{\max_{0\leq t<T} \rho_t((-\infty,0])}$$

$$p^+ := \frac{\max_{0\leq t<T} \rho_t([0,\infty)) - \min_{0\leq t<T} \rho_t([0,\infty))}{\max_{0\leq t<T} \rho_t([0,\infty))},$$

are lower bounds for the probability for a particle to move from the left to the right well or viceversa

NB In the case of (1), $p^- = p^+$.

Comparison for different values of σ between the indicator p and the asymptotic amplitude $\bar{x}$ [2].



[2] L. Gammaitoni et al., Reviews of Modern Physics (1998)

Thanks!